# INTEGRABLE SYSTEMS, TORIC DEGENERATIONS AND OKOUNKOV BODIES

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Abstract. Let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$ equipped with a very ample line bundle $\mathcal{L}$. Using the theory of Okounkov bodies and an associated toric degeneration, we construct – under a mild technical hypothesis on $X$ – an *integrable system* on $X$ in the sense of symplectic geometry. More precisely, we construct a collection of real-valued functions $\{H_1, \ldots, H_n\}$ on $X$ which are continuous on all of $X$, smooth on an open dense subset $U$ of $X$, and pairwise Poisson-commute on $U$. Moreover $\{H_1, \ldots H_n\}$ generate a torus action on $U$. Here the symplectic structure on $X$ is the pullback of the Fubini-Study form on $\mathbb{P}(H^0(X, \mathcal{L})^*)$ via the Kodaira embedding. The image of the 'moment map' $\mu = (H_1, \ldots, H_n) : X \to \mathbb{R}^n$ is precisely the *Okounkov body* $\Delta = \Delta(R, v)$ associated to the homogeneous coordinate ring $R$ of $X$, and an appropriate choice of valuation $v$ on $R$. Our main technical tools come from algebraic geometry, differential (Kähler) geometry, and analysis. Specifically, we use: a toric degeneration of $X$ to a (not necessarily normal) toric variety $X_0$, the gradient-Hamiltonian vector field, and a subtle generalization of the famous Łojasiewicz gradient inequality for real-valued analytic functions. Since our construction is valid for a large class of projective varieties $X$, this manuscript provides a rich source of new examples of integrable systems. We discuss concrete examples, including elliptic curves, flag varieties of arbitrary connected complex reductive groups, spherical varieties, and weight varieties. In future work we intend to investigate further all of the concrete examples mentioned here, as well as the moduli spaces of flat connections on Riemann surfaces.

*This is a preliminary version. Comments are welcome.*

## Contents



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## 1. Introduction

A (completely) integrable system on a symplectic manifold is a Hamiltonian system which admits a maximal number of first integrals, also called 'conservation laws'. A first integral is a function which is constant along the Hamiltonian flow; when there are a maximal number of such, then one can describe the integral curves of the Hamiltonian vector field implicitly by setting the first integrals equal to constants. In this sense an integrable system is very well-behaved. For a recent overview of this subject, see [PeVu11] and its extensive bibliography. The theory of integrable systems in symplectic geometry is rather dominated by specific examples (e.g. 'spinning top', 'Calogero-Moser system', 'Toda lattice'). The main contribution of this manuscript, summarized in Theorem 1.3 below, is a construction of an integrable system on (an open dense subset of) a wide class of complex projective varieties $X$. Our methods therefore substantially contributes to the set of known examples, with a corresponding expansion of the possible applications of integrable systems theory to other research areas, including: algebraic geometry, symplectic topology, representation theory, and Schubert calculus, to name just a few. We therefore expect our manuscript to be of wide-ranging interest. Moreover, we use a rather broad mix of techniques coming from algebraic geometry, differential (Kähler) geometry, and analysis, in order to prove our main results. With this in mind, we have attempted to make our exposition accessible to a broad audience.

We begin with a definition; for details see e.g. [Au96]. Let $(M, \omega)$ be a symplectic manifold of real dimension $2n$. Let $\{f_1, f_2, \ldots, f_n\}$ be smooth functions on $M$. Then the functions $\{f_1, \ldots, f_n\}$ are called an *integrable system* on $M$ if they satisfy the following conditions:

- they pairwise *Poisson-commute*, i.e. $\{f_i, f_j\} = 0$ for all $i, j$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket on $\mathcal{C}^\infty(M)$ induced from $\omega$, and
- they are *functionally independent*, i.e. their derivatives $df_1, \ldots, df_n$ are linearly independent almost everywhere on $M$.

In fact, we will later state and use a slightly generalized version of this definition, cf. Definition 5.1.

We recall two examples which may be familiar to researchers outside of symplectic geometry.

**Example 1.1.** A smooth projective toric variety $X$ is a symplectic manifold, equipped with the pullback of the standard Fubini-Study form on projective space. The (compact) torus action on $X$ is in fact Hamiltonian in the sense of symplectic geometry, and its moment map image is precisely the polytope corresponding to $X$. The torus has real dimension $n = \frac{1}{2}\dim_{\mathbb{R}}(X)$, and its $n$ components form an integrable system on $X$.

**Example 1.2.** Let $X = \mathrm{GL}(n, \mathbb{C})/B$ be the flag variety of nested subspaces in $\mathbb{C}^n$. For $\lambda$ a regular highest weight, consider the usual Plücker embedding $X \hookrightarrow \mathbb{P}(V_\lambda)$ where $V_\lambda$ denotes the irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with highest weight $\lambda$. Equip $X$ with the Kostant-Kirillov-Souriau symplectic form coming from its identification with a coadjoint orbit of $U(n, \mathbb{C})$, which meets the positive Weyl chamber at precisely $\lambda$. Then Guillemin-Sternberg [GuSt83] build an integrable system, frequently called the *Gel'fand-Cetlin integrable system*, on $X$ by viewing this coadjoint orbit as a subset of the space of Hermitian $n \times n$ matrices. This integrable system is intimately related to the well-known Gel'fand-Cetlin canonical basis for the irreducible representation $V_\lambda$ [GeCe50].

Example 1.2 in fact motivated this paper. Several years ago, we had the idea to use the toric degeneration results for flag varieties given in Gonciulea-Lakshmibai [GoLa96], Kogan-Miller [KoMi05], and Caldero [Ca02], in order to construct integrable systems for flag varieties of general reductive groups, thus generalizing the Gel'fand-Cetlin integrable system constructed by Guillemin and Sternberg. Allen Knutson also had very similar ideas, according to his post on MathOverflow in response to a question from David Treumann [KnTr-MO]. (His post is dated January 2010, but he apparently had the ideas long before.) More formally, a connection between toric degenerations and the Gel'fand-Cetlin integrable system was established in detail in 2010 in [NiNoUe10], using Ruan's technique of gradient-Hamiltonian flows. Their ideas inspired us to prove the results contained in the present paper, which in fact deals not only with flag varieties of general reductive groups, but with a much more general class of projective varieties. To do this, we use the theory of **Okounkov bodies**, as we briefly explain now. This theory was initiated by Okounkov [Ok96, Ok00] and developed by Kaveh and Khovanskii [KaKh-NO] (also independently by Lazarsfeld and Mustata [LaMu09]).

Suppose $X$ is a complex projective variety equipped with a very ample line bundle $\mathcal{L}$ and let $R$ denote the corresponding homogeneous coordinate ring. Let $n = \dim_{\mathbb{C}}(X)$. Let $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ be a valuation with one-dimensional leaves (Definition 2.1) on the space of rational functions $\mathbb{C}(X)$ on $X$, and let $S = S(R)$ denote the corresponding value semigroup (see equation (2.2)) in $\mathbb{N} \times \mathbb{Z}^n$. We also let $\Delta = \Delta(R)$ denote the **Okounkov body** corresponding to $R$ and $v$ (Definition 2.7). With this notation in place we may state the main result of this manuscript (Theorem 5.20).

**Theorem 1.3.** *In the setting above, suppose that $X$ is smooth and that $S$ is a finitely generated semigroup (this implies that $\Delta$ is a rational polytope). Then there exist functions $\{f_1, \ldots, f_n\}$ on $X$ such that*

- *the $f_i$ are continuous on all of $X$, and smooth on an open dense subset $U$ of $X$,*
- *the $f_i$ pairwise Poisson-commute on $U$,*
- *moreover, the functions $(f_1, \ldots, f_n)$ generate a torus action on $U$, and*
- *the image of $X$ under $\mu := (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is precisely the Okounkov body $\Delta$.*

In particular, our theorem addresses a question posed to us by Julius Ross and by Steve Zelditch: does there exist, in general, a 'reasonable' map from a (smooth) variety $X$ to its Okounkov body? At least under the technical assumption that the value semigroup $S$ is finitely generated, our theorem suggests that the answer is yes. Furthermore, we expect that the methods of this manuscript can be modified to prove a similar statement for the case when the value semigroup $S$ is not finitely generated.

We now briefly sketch the idea of our proof. The essential ingredient is the toric degeneration from $X$ to a toric variety $X_0$ (the normalization of which is the toric variety corresponding to the polytope $\Delta = \Delta(R)$), discussed in detail by Anderson [An10]. (In fact, the existence of a toric degeneration from an algebra to its semigroup algebra associated to a valuation is proven already by Teissier in [Te99].) Let $\pi : \mathcal{X} \to \mathbb{C}$ denote the flat family with special fiber $\pi^{-1}(0) \cong X_0$ and $\pi^{-1}(z) = X_z \cong X$ for $z \neq 0$. Since toric varieties are integrable systems (see example above), the idea is to "pull back" the integrable system on $X_0$ to one on $X$. To accomplish this we use the so-called 'gradient Hamiltonian vector field' (first defined by Ruan [Ru99] and also used by Nishinou-Nohara-Ueno [NiNoUe10]) on $\mathcal{X}$, where we think of $\mathcal{X}$ as a symplectic space by embedding it into the product of an appropriate projective space with $\mathbb{C}$. The main technicalities which must be overcome in order to make this sketch rigorous is to appropriately deal with the singular points of $\mathcal{X}$

and prove that the $f_i$ thus constructed may be continuously extended to all of $X$. It turns out that, in order to deal with this issue, we need a subtle generalization of the famous Łojasiewicz inequality.

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## 2. Preliminaries

2.1. **General setup.** Let $X$ be a projective variety of dimension $n$ over $\mathbb{C}$ equipped with a very ample line bundle $\mathcal{L}$. Let $L := H^0(X, \mathcal{L})$ denote the space of global sections of $\mathcal{L}$; it is a finite dimensional vector space over $\mathbb{C}$. The line bundle $\mathcal{L}$ gives rise to the *Kodaira map* $\Phi_L$ *of* $L$, from $X$ to the projective space $\mathbb{P}(L^*)$, defined as follows: the image $\Phi_L(x)$ of a point $x \in X$ is the point in $\mathbb{P}(L^*)$ corresponding to the hyperplane

$$H_x = \{f \in L \mid f(x) = 0\} \subset L.$$

The assumption that $\mathcal{L}$ is very ample implies that the Kodaira map $\Phi_L$ is an embedding.

Alternatively we can describe $\Phi_L$ more concretely as follows. Let $x \in X$. Let $h \in L$ be a section of $\mathcal{L}$ with $h(x) \neq 0$; such an $h$ exists since $\mathcal{L}$ is very ample (hence $L$ has no base locus). Let $\ell_x \in L^*$ be defined by:

$$\ell_x(f) = f(x)/h(x), \quad \forall f \in L.$$

Then $\Phi_L(x)$ is the point in $\mathbb{P}(L^*)$ represented by $\ell_x$. It is straightforward to check that $\Phi_L(x)$ is independent of the choice of the section $h$.

Now let $L^k$ denote the image of the $k$-fold product $L \times \cdots \times L$ in $H^0(X, \mathcal{L}^{\otimes k})$ under the natural map given by taking the product of sections. (In general this map may not be surjective.) The homogeneous coordinate ring of $X$ with respect to the embedding $\Phi_L : X \hookrightarrow \mathbb{P}(L^*)$ can be identified with the graded algebra

$$R = R(L) = \bigoplus_{k \geq 0} R_k,$$

where $R_k := L^k$. This is a subalgebra of the *ring of sections*

$$R(\mathcal{L}) = \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k}).$$

Recall that the *Hilbert function* of the graded algebra $R$ is defined as $H_R(k) := \dim_{\mathbb{C}}(R_k) = \dim_{\mathbb{C}}(L^k)$. The celebrated theorem of Hilbert on the degree of a projective variety states the following:

(1) For sufficiently large values of $k$, the function $H_R(k)$ coincides with a polynomial of degree $n = \dim_{\mathbb{C}}(X)$.
(2) Let $a_n := \lim_{k \to \infty} H_R(k)/k^n$ be the leading coefficient of this polynomial. Then $n!a_n$ is equal to the degree of the projective embedding of $X$ in $\mathbb{P}(L^*)$ (in other words, the self-intersection number of the divisor class of the line bundle $\mathcal{L}$).

2.2. **Valuations and Okounkov bodies.** Okounkov bodies give information about asymptotic behavior of Hilbert functions of graded algebras. In this section we review some background material about Okounkov bodies following [KaKh08, KaKh-NO]. In the present paper we will only be concerned with graded algebras which are homogeneous coordinate rings of projective varieties, as introduced in the previous section, so we restrict to this case in the discussion below.

Following the notation of Section 2.1, we now wish to associate a convex body $\Delta(R) \subset \mathbb{R}^n$ to $R$. In the case when $X$ is a projective toric variety, there is a well-known such convex body - namely, the Newton polytope of $X$. However, for a general projective variety, we do not have such toric methods at our disposal. The tool we use for this purpose - as initially proposed by Okounkov in [Ok96, Ok00] and further developed in [KaKh08, KaKh-NO, LaMu09] - is a **valuation** on the field $\mathbb{C}(X)$ of rational functions on the variety $X$. We now recall the definition.

We fix once and for all a total order $<$ on the lattice $\mathbb{Z}^n$ which respects addition. Indeed, for additional concreteness, we always consider the standard lexicographic order.

**Definition 2.1.** A *valuation* on the field $\mathbb{C}(X)$ is a function $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ satisfying the following: for any $f, g \in \mathbb{C}(X) \setminus \{0\}$,

(a) $v(fg) = v(f) + v(g)$,
(b) $v(f + g) \geq \min(v(f), v(g))$, and
(c) $v(\lambda f) = v(f)$, for all $0 \neq \lambda \in \mathbb{C}$.

Moreover, we say the valuation $v$ has *one-dimensional leaves* if it additionally satisfies the following:

$$\text{if } v(f) = v(g), \text{ then there exists a non-zero constant } \lambda \neq 0 \in \mathbb{C} \\ \text{such that } v(g - \lambda f) > v(g) \text{ or } g - \lambda f = 0. \tag{2.1}$$

(In the literature, such a valuation is also sometimes called a *zero-dimensional valuation*, since this condition implies that the residue field of $v$ is $\mathbb{C}$.)

If $v$ is a valuation with one-dimensional leaves, then the image of $v$ is a sublattice of $\mathbb{Z}^n$ of full rank. Hence, by replacing $\mathbb{Z}^n$ with this sublattice if necessary, we will always assume without loss of generality that $v$ is surjective.

Given a variety $X$, there exist many possible valuations with one-dimensional leaves on its field of rational functions $\mathbb{C}(X)$. Below, we list several examples which arise quite naturally in geometric contexts.

**Example 2.2.** Let $X$ be a curve and let $p$ be a smooth point on $X$. Then

$$v(f) := \text{ order of zero/pole of } f \text{ at } p$$

defines a valuation. More generally, if $X$ is an $n$-dimensional variety, a choice of a coordinate system at a smooth point $p$ on $X$ gives a valuation on $\mathbb{C}(X)$ with one-dimensional leaves.

**Example 2.3.** Generalizing Example 2.2 further, we can construct a valuation out of a flag of subvarieties in $X$. More specifically, let

$$\{p\} = X_n \subset \cdots \subset X_0 = X$$

be a sequence of closed irreducible subvarieties of $X$ such that $\dim_{\mathbb{C}}(X_k) = n - k$ for $0 \leq k \leq n$, and assume $X_k$ is a normal subvariety of $X_{k-1}$ for $0 < k \leq n$. Such a sequence is called a *sequence of normal subvarieties* (or a *normal Parshin point*) on the variety $X$. A collection $u_1, \ldots, u_n$ of rational functions on $X$ is a *system of parameters* about such a sequence if for each $k$, $u_{k|X_k}$ is a rational function on $X_k$ which is not identically zero and

which has a zero of first order on the hypersurface $X_{k+1}$. By the normality assumption above, such a system of parameters always exists. Given a sequence of normal subvarieties and a system of parameters $u_1, \ldots, u_n$, we can define a valuation $v$ on $\mathbb{C}(X)$ with one-dimensional leaves and values in $\mathbb{Z}^n$ as follows. Let $f \in \mathbb{C}(X)$ with $f \neq 0$. Then $v(f) = (k_1, \ldots, k_n)$ where the $k_i$ are defined inductively as follows. The first coordinate $k_1$ is the order of vanishing of $f$ on $X_1$. Then $f_1 = (u_1^{-k_1} f)_{|X_1}$ is a well-defined rational function on $X_1$ which is not identically zero, and $k_2$ is the order of vanishing of $f_1$ on $X_2$. Continuing in this manner defines all $k_i$. (In fact, the normality assumption on the $X_k$ is not crucial; it can be avoided by passing to the normalizations.)

The following proposition is simple but fundamental; it states that the image of the valuation on a finite-dimensional subspace $E$ of $\mathbb{C}(X)$ is in one-to-one correspondence with a basis of $E$, in much the same way that the integral points in the Newton polytope of a projective toric variety $X$ correspond to a basis of $H^0(X, \mathcal{L})$. The proof is straightforward from the defining properties of a valuation with one-dimensional leaves.

**Proposition 2.4.** *Let $E \subset \mathbb{C}(X)$ be a finite-dimensional subspace of $\mathbb{C}(X)$. Then $\dim_{\mathbb{C}} E = \#v(E \setminus \{0\})$.*

Fix a choice of valuation $v$ with one-dimensional leaves on $\mathbb{C}(X)$. Using the valuation $v$ we now associate a semigroup $S(R) \subset \mathbb{N} \times \mathbb{Z}^n$ to the homogeneous coordinate ring $R$ of $X$. First we identify $L = H^0(X, \mathcal{L})$ with a (finite-dimensional) subspace of $\mathbb{C}(X)$ by choosing a non-zero element $h \in L$ and mapping $f \in L$ to the rational function $f/h \in \mathbb{C}(X)$. Similarly, we can associate the rational function $f/h^k$ to an element $f \in R_k := L^k \subseteq H^0(X, \mathcal{L}^{\otimes k})$. Using these identifications, we define

$$S = S(R) = S(R, v, h) = \bigcup_{k>0} \{(k, v(f/h^k)) \mid f \in L^k \setminus \{0\}\}. \tag{2.2}$$

From the property (a) in Definition 2.1 it follows that $S(R)$ is an additive semigroup. Moreover, from the property (2.1) in Definition 2.1 it is straightforward to show the following.

**Proposition 2.5.** *The group generated by the semigroup $S = S(R)$, considered as a semigroup in $\mathbb{Z} \times \mathbb{Z}^n \cong \mathbb{Z}^{n+1}$, is (all of) $\mathbb{Z}^{n+1}$.*

**Remark 2.6.** The semigroup $S = S(R)$ depends on the choice of valuation $v$ on $\mathbb{C}(X)$ and the section $h$. The dependence on $h$ is minor; a different choice of $h'$ would lead to a semigroup which is shifted by the vector $kv(h/h')$ at the level $\{k\} \times \mathbb{Z}^n$. However, the dependence on the valuation $v$ is much more subtle.

In order to keep track of the natural $\mathbb{N}$-grading on the ring $R = \oplus_{k \geq 0} R_k$, it is convenient to extend the valuation $v$ to a valuation $\tilde{v} : R \setminus \{0\} \to \mathbb{N} \times \mathbb{Z}^n$ as follows. We define an ordering on $\mathbb{N} \times \mathbb{Z}^n$ by $(m, u) \leq (m', u')$ if and only if

$$\text{either } (m > m') \quad \text{or} \quad (m = m' \text{ and } u \leq u') \text{ (note the switch!)} \tag{2.3}$$

For $f \in R$, we now define

$$\tilde{v}(f) := (m, v(f_m/h^m)) \tag{2.4}$$

where $f_m$ is the highest-degree homogeneous component of $f$. The map $\tilde{v}$ is a valuation on $R$ with the above ordering on $\mathbb{N} \times \mathbb{Z}^n$. By construction, the image of the valuation $\tilde{v}$ is exactly the semigroup $S = S(R)$.

Given $S \subset \mathbb{N} \times \mathbb{Z}^n$ an arbitrary additive semigroup, we can associate to it a convex body as follows. Let $C(S)$ denote the closure of the convex hull of $S \cup \{0\}$, considered as a subset

of $\mathbb{R}_{\geq 0} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$. It is a closed convex cone with apex at the origin. If $S$ is a finitely generated semigroup then $C(S)$ is a polyhedral cone. Consider the intersection of the cone $C(S)$ with the plane $\{1\} \times \mathbb{R}^n$ and let $\Delta(S)$ denote the projection of this intersection to $\mathbb{R}^n$, via the projection on the second factor $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Equivalently, $\Delta(S)$ can be described as

$$\Delta(S) = \overline{\mathrm{conv}(\bigcup_{k>0} \{x/k \mid (k,x) \in S\})}. \tag{2.5}$$

If the cone $C(S)$ intersects the plane $\{0\} \times \mathbb{R}^n$ only at the origin, then the convex set $\Delta(S)$ is bounded and hence is a convex body. This will be the case for all the semigroups we consider in this paper. We refer to $\Delta(S)$ as the **convex body associated to the semigroup $S$.**

**Definition 2.7.** Let $S = S(R)$ be the semigroup associated to $(R, v, h)$ as above. We denote the cone $C(S)$ and the convex body $\Delta(S)$ by $C(R)$ and $\Delta(R)$ respectively. In this case, the cone $C(R)$ intersects $\{0\} \times \mathbb{R}^n$ only at the origin and hence $\Delta(R) = \Delta(R, v, h)$ is a convex body, the **Okounkov body of** $(R, v, h)$. It is also sometimes denoted as $\Delta(X, v)$ or simply $\Delta(X)$ when we wish to emphasize the underlying projective variety $X$.

**Remark 2.8.** From Remark 2.6 it follows that a different choice $h' \in L$ would yield an Okounkov body $\Delta(X, v, h')$ which is shifted by the fixed vector $v(h/h')$. Thus, similar to the case of $S(R)$, the dependence of the Okounkov body on the choice of section $h$ is minor. However, as in Remark 2.6, the dependence of $\Delta(X, v)$ on the valuation $v$ is more subtle.

The Okounkov body $\Delta(R)$ encodes information about the asymptotic behavior of the Hilbert function of $R$ (see [KaKh08, KaKh-NO], [LaMu09] and [Ok96, Ok00]). Let $H_R(k) := \dim_{\mathbb{C}}(R_k)$ be the Hilbert function of the graded algebra $R$.

**Theorem 2.9.** *The Okounkov body $\Delta(R)$ has real dimension $n$, and the leading coefficient*

$$a_n = \lim_{k\to\infty} \frac{H_R(k)}{k^n},$$

*of the Hilbert function of $R$ is equal to $\mathrm{Vol}_n(\Delta(R))$, the Euclidean volume of $\Delta(R)$ in $\mathbb{R}^n$. In particular, the degree of the projective embedding of $X$ in $\mathbb{P}(L^*)$ is equal to $n!\mathrm{Vol}_n(\Delta(R))$.*

**Remark 2.10.** In [KaKh-NO], the asymptotic behavior of Hilbert functions of a much more general class of graded algebras is addressed, namely, graded algebras of the form $A = \bigoplus_{k\geq 0} A_k$ such that

- $A_k \subset \mathbb{C}(X)$ for all $k$, and
- $A$ is contained in a finitely generated algebra of the same form.

Note that such an algebra $A$ is not necessarily finitely generated. This class of graded algebras includes arbitrary *graded linear series* on a variety, and in particular, the ring of sections of arbitrary line bundles (not just very ample ones).

## 3. Weighted projective space

The constructions in the following sections require us to work with weighted projective spaces. In this section we address some general facts and constructions regarding these generalizations of classical projective spaces.

Let $V_1, \ldots, V_r$ be finite dimensional vector spaces over $\mathbb{C}$ and $m_1, \ldots, m_r$ positive integers. Consider the action of $\mathbb{C}^*$ on $V_1 \times \cdots \times V_r$ by:

$$\lambda \cdot (v_1, \ldots, v_r) = (\lambda^{m_1} v_1, \ldots, \lambda^{m_r} v_r).$$

We define the *weighted projective space* $W\mathbb{P} = W\mathbb{P}(V_1, \dots, V_r; m_1, \dots, m_r)$ to be the quotient of $(V_1 \times \cdots \times V_r) \setminus \{(0, \dots, 0)\}$ by this $\mathbb{C}^*$-action. The point in $W\mathbb{P}$ represented by $(v_1, \dots, v_r) \in V_1 \times \cdots \times V_r$ is denoted $(v_1 : \cdots : v_r)$. There is a natural rational map $W\mathbb{P} \dashrightarrow \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_r)$ given by:

$$(v_1 : \cdots : v_r) \mapsto ([v_1], \dots, [v_r]).$$

This is defined whenever all the $v_i$ are nonzero. The following is well-known.

**Theorem 3.1.** *The weighted projective space $W\mathbb{P}$ is a normal projective variety. Moreover, $W\mathbb{P}$ is smooth if and only if $m_1 = \cdots = m_r$.*

3.1. **Embedding a weighted projective space in a large projective space.** In the constructions that follow, it will be useful to embed the (usually singular) weighted projective space into a larger, classical (and hence smooth) projective space. In this section we describe in detail a particular such embedding which we use repeatedly throughout this paper.

Let $\mathrm{Sym}((V_1 \times \cdots \times V_r)^*)$ denote the algebra of polynomials on $V_1 \times \cdots \times V_r$. We equip $\mathrm{Sym}((V_1 \times \cdots \times V_r)^*) \cong \mathrm{Sym}(V_1^* \times V_2^* \times \cdots \times V_r^*)$ with a grading by assigning for any $i$, $1 \le i \le r$, and any $f_i^* \in V_i^*$, the degree $m_i$. Let $d > 0$ be a positive integer. With respect to the above grading, let $\mathbf{V}_d$ denote the subspace of $\mathrm{Sym}((V_1 \times \cdots \times V_r)^*)$ consisting of the elements of total degree $d$. From the definition of the grading it then follows that

$$\mathbf{V}_d \cong \bigoplus_{\substack{\beta=(\beta_1,\dots,\beta_r)\in\mathbb{Z}^r_{\ge 0} \\ \sum_i m_i\beta_i = d}} \bigotimes_{i=1}^{r} \mathrm{Sym}_{\beta_i}(V_i^*) \tag{3.1}$$

where $\mathrm{Sym}_{\beta_i}(V_i^*)$ denotes the space of homogeneous polynomials of degree $\beta_i$ on $V_i$.

Given $v = (v_1, \dots, v_r) \in (V_1 \times \cdots \times V_r)$, the evaluation map

$$\ell_v : p \mapsto p(v)$$

defines a linear function $\ell_v$ on $\mathrm{Sym}((V_1 \times \cdots \times V_r)^*)$. It follows that the map $v \mapsto \ell_v|_{\mathbf{V}_d}$ induces a rational map

$$\Theta : W\mathbb{P}(V_1, \dots, V_r) \dashrightarrow \mathbb{P}(\mathbf{V}_d^*).$$

Moreover, we have the following [Do82].

**Proposition 3.2.** *If $d = \prod_{i=1}^r m_i$ then $\Theta$ is a morphism and an embedding.*

The map $\Theta$ can also be described concretely in coordinates. Suppose $n_i = \dim_{\mathbb{C}}(V_i)$ for each $i$ and fix a basis for each $V_i$; let $(x_{i1}, \dots, x_{in_i})$ denote the corresponding coordinates for $V_i$ (i.e. dual vectors in $V_i^*$). Then, under the identification $\mathrm{Sym}((V_1 \times \cdots \times V_r)^*) \cong \mathrm{Sym}(V_1^* \times \cdots \times V_r^*) \cong \mathbb{C}[x_{ij}; 1 \le i \le r, 1 \le j \le n_i]$, the set of monomials

$$\left\{ x_{11}^{\alpha_{11}} \cdots x_{rn_r}^{\alpha_{rn_r}} \,\middle|\, \sum_{i=1}^{r} m_i \sum_{j=1}^{n_i} \alpha_{ij} = d,\ \alpha_{ij} \in \mathbb{Z}_{\ge 0} \right\} \tag{3.2}$$

form a basis for $\mathbf{V}_d$. With respect to these coordinates, the morphism $\Theta : W\mathbb{P} \to \mathbb{P}(\mathbf{V}_d^*)$ can be described explicitly as

(3.3)

$$\Theta : ((x_{11}, \dots, x_{1n_1}) : \cdots : (x_{r1}, \dots, x_{rn_r})) \mapsto \left( x_{11}^{\alpha_{11}} \cdots x_{rn_r}^{\alpha_{rn_r}} \,\middle|\, \sum_{i=1}^{r}\sum_{j=1}^{n_i} m_i \alpha_{ij} = d,\ \alpha_{ij} \in \mathbb{Z}_{\ge 0} \right).$$

The following will be used in Section 6.

**Remark 3.3.** Let $\mathbb{T} \cong (\mathbb{C}^*)^m$ be an algebraic torus. Suppose $\mathbb{T}$ acts linearly on each $V_i$. The diagonal $\mathbb{T}$-action on $V_1 \times \cdots \times V_r$ induces an action of $\mathbb{T}$ on $W\mathbb{P}(V_1, \ldots, V_r; m_1, \ldots, m_r)$ and on $\mathbb{P}(\mathbf{V}_d^*)$. It is not hard to see that $\Theta$ is $\mathbb{T}$-equivariant with respect to these actions.

3.2. **Embedding a projective variety $X$ into a weighted projective space.** Returning now to the general setup of Section 2.1, let $X$ be a projective variety equipped with a very ample line bundle $\mathcal{L}$ and let $L = H^0(X, \mathcal{L})$. Let $r > 0$ be a positive integer. For any $k$ with $1 \leq k \leq r$, define $V_k := (L^k)^*$ and $m_k := k$. The construction outlined above yields the associated weighted projective space

$$W\mathbb{P} = W\mathbb{P}(L^*, (L^2)^*, \ldots, (L^r)^*; 1, 2, \ldots, r).$$

Analogous to the construction of the Kodaira map $\Phi_L$, we can also define a morphism $\Psi : X \to W\mathbb{P}$ as follows. Let $x \in X$. Let $h \in L$ be a section with $h(x) \neq 0$; as before, since $L$ has no base locus, such an $h$ always exists. For each $k$ with $1 \leq k \leq r$, let $\ell_{x,k}$ be the element in $(L^k)^*$ defined by

$$\ell_{x,k}(f) := f(x)/h^k(x), \quad \forall f \in L^k.$$

Define $\Psi(x)$ to be the point $(\ell_{x,1} : \cdots : \ell_{x,r}) \in W\mathbb{P}$. It can be checked that $\Psi(x)$ is independent of the choice of the section $h$ and is a morphism from $X$ to $W\mathbb{P}$. Note that each $\Phi_{L^k}$ is an embedding, so $\Psi$ is also an embedding.

The morphism $\Psi$ can also be described explicitly in terms of coordinates (at least on an open dense subset) as follows. Fix bases $\{f_{i1}, \ldots, f_{in_i}\}$ for the $L^i$. With respect to the corresponding dual bases for the $(L^i)^*$), the map $\Psi$ can be given in coordinates by

$$\Psi : x \mapsto \left( \left( \frac{f_{11}(x)}{h(x)}, \ldots, \frac{f_{1n_1}(x)}{h(x)} \right) : \cdots : \left( \frac{f_{r1}(x)}{h^r(x)}, \ldots, \frac{f_{rn_r}(x)}{h^r(x)} \right) \right) \tag{3.4}$$

at the points $x \in X$ where $h(x) \neq 0$.

Following Section 3.1, let $\mathbf{V}_d$ denote the subspace of $\mathrm{Sym}(L \times \cdots \times L^r) \cong \mathrm{Sym}((L^* \times \cdots \times (L^r)^*)^*)$ consisting of homogeneous elements of degree $d$ (recall that $\deg(f_i) = i$ for $f_i \in L^i$ in this grading). Let $d = r!$. Proposition 3.2 then implies that the composition $\Theta \circ \Psi : X \to \mathbb{P}(\mathbf{V}_d^*)$ is an embedding. Again we take a moment to describe $\Theta \circ \Psi$ explicitly in terms of coordinates. As before, let $x \in X$ and let $h \in L$ such that $h(x) \neq 0$. Using the same bases $\{f_{i1}, \ldots, f_{in_i}\}$ for $L^i$ as above, note that any $p \in \mathbf{V}_d$ can be written as

$$p = \sum_{\substack{\alpha = (\alpha_{11}, \ldots, \alpha_{1n_1}, \ldots, \alpha_{r1}, \ldots, \alpha_{rn_r}), \alpha_{ij} \in \mathbb{Z}_{\geq 0} \\ \sum_{i=1}^r i(\sum_{j=1}^{n_i} \alpha_{ij}) = d = r!}} c_\alpha f_{11}^{\alpha_{11}} \cdots f_{1n_1}^{\alpha_{1n_1}} \cdots f_{r1}^{\alpha_{r1}} \cdots f_{rn_r}^{\alpha_{rn_r}}.$$

From the previous coordinate descriptions of $\Theta$ and $\Psi$ it is now straightforward to see that $(\Theta \circ \Psi)(x) \in \mathbb{P}(\mathbf{V}_d^*)$ can be described as the point

$$\left( \frac{f_{11}(x)^{\alpha_{11}} \cdots f_{rn_r}(x)^{\alpha_{rn_r}}}{h(x)^d} \;\middle|\; \sum_{i=1} i \left( \sum_{j=1}^{n_i} \alpha_{ij} \right) = d = r! \;,\; \alpha_{ij} \in \mathbb{Z}_{\geq 0} \right) \tag{3.5}$$

on the locus where $h(x) \neq 0$.

For each $i > 0$, the sum and product of sections give a natural linear map from $L^{\otimes i}$ onto $L^i$. Similarly there is a natural linear map from $\mathbf{V}_d$ onto $L^d$. These induce embeddings $(L^i)^* \hookrightarrow (L^{\otimes i})^*$ and $(L^d)^* \hookrightarrow \mathbf{V}_d^*$, which in turn give embeddings of the corresponding

projective spaces $\mathbb{P}((L^i)^*) \hookrightarrow \mathbb{P}((L^{\otimes i})^*)$ and $j : \mathbb{P}((L^d)^*) \hookrightarrow \mathbb{P}(\mathbf{V}_d^*)$. From (3.5) and the definition of the Kodaira map $\Phi_{L^d}$, it follows that the diagram

$$\begin{tikzcd} X \arrow[r, "\Psi"] \arrow[dr, "\Phi_{L^d}"'] & W\mathbb{P} \arrow[r, "\Theta"] & \mathbb{P}(\mathbf{V}_d^*) \\ & \mathbb{P}((L^d)^*) \arrow[ur, "j"'] & \end{tikzcd} \tag{3.6}$$

commutes.

The following will be used in Section 6.

**Remark 3.4.** Suppose $\mathbb{T}$ is an algebraic torus. Suppose $\mathbb{T}$ acts on $X$ and that the very ample line bundle $\mathcal{L}$ is $\mathbb{T}$-linearized. Then, for any $k > 0$, the spaces $(L^k)^*$ are $\mathbb{T}$-modules and hence $\mathbb{T}$ also acts on $W\mathbb{P}$ and $\mathbb{P}(\mathbf{V}_d^*)$ as in Remark 3.3. It can be checked that all the maps in the diagram (3.6) are $\mathbb{T}$-equivariant for the corresponding $\mathbb{T}$-actions.

3.3. **Kähler structures.** We fix for the remainder of this discussion a Hermitian product $H$ on $L^*$. Such a Hermitian product gives rise to a Kähler structure $\Omega_H$ on $\mathbb{P}(L^*)$ (see e.g. [GrHa78]). Let $\omega$ denote the pull-back of $\Omega_H$ to $X$ under the Kodaira map $\Phi_L$. In this section we wish to construct a Kähler structure on the (smooth locus of the) weighted projective space $W\mathbb{P}$ such that its pull-back under the embedding $\Psi$ coincides with the form $\omega$ on $X$. We will use this construction to give a Kähler structure to the degenerating family $\mathfrak{X}$ which we construct in Section 4.

We will obtain the required Kähler structure from a Kähler structure on the projective space $\mathbb{P}(\mathbf{V}_d^*)$. For this purpose, it is convenient to extend the diagram (3.6). We maintain the notation of Sections 3.1 and 3.2. Suppose $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r_{\geq 0}$ where $\sum_i i \cdot \beta_i = d$, as in Section 3.1. Identifying $L^{\otimes d} \cong \otimes_{i=1}^r L^{\otimes i \cdot \beta_i}$ in the natural way, consider the map

$$L^{\otimes d} \to \otimes_{i=1}^r \mathrm{Sym}_{\beta_i}(L^i) \tag{3.7}$$

induced by taking the natural map $L^{\otimes i \cdot \beta_i} \to \mathrm{Sym}_{\beta_i}(L^i)$, induced by the product of sections, on each factor. Taking the direct sum of (3.7) over all such $\beta$ we obtain a natural map

$$\bigoplus_{\substack{\beta=(\beta_1,\ldots,\beta_r)\in\mathbb{Z}^r_{\geq 0} \\ \sum_{i=1}^r i\cdot\beta_i = d}} L^{\otimes d} \to \mathbf{V}_d \tag{3.8}$$

using the identification (3.1). Since (3.8) is induced from taking the product of sections, it is straightforward to see that (3.8) fits into a commutative diagram

$$\begin{tikzcd} \bigoplus_{\substack{\beta=(\beta_1,\ldots,\beta_r)\in\mathbb{Z}^r_{\geq 0} \\ \sum_{i=1}^r i\cdot\beta_i = d}} L^{\otimes d} \arrow[r] \arrow[d] & \mathbf{V}_d \arrow[d] \\ L^{\otimes d} \arrow[r] & L^d. \end{tikzcd}$$

where the left vertical map takes the sum of the components in the direct sum, the top horizontal arrow is (3.8), and the other two arrows are the natural ones induced from the sum and product of sections. It is clear that all four maps are surjective.

Taking duals, we get the diagram of inclusions

$$
\begin{tikzcd}
\mathbf{V}_d^* \arrow[r, hook] & \bigoplus_{\substack{\beta=(\beta_1,\dots,\beta_r)\in\mathbb{Z}^r_{\geq 0}\\ \sum_{i=1}^r i\cdot\beta_i=d}} (L^{\otimes d})^* \\
(L^d)^* \arrow[u, hook] \arrow[r, hook] & (L^{\otimes d})^* \arrow[u, hook]
\end{tikzcd}
\tag{3.9}
$$

where the right vertical map is given by the diagonal inclusion $v \mapsto (v, v, \dots, v)$. Considering the induced morphisms between the corresponding projective spaces and putting this together with (3.6), we obtain the commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "\Psi"] \arrow[d, "\Phi_L"] \arrow[drr, "\Phi_{L^d}"'] & W\mathbb{P} \arrow[r, "\Theta"] & \mathbb{P}(\mathbf{V}_d^*) \arrow[r] & \mathbb{P}(\bigoplus_\beta (L^{\otimes d})^*) \\
\mathbb{P}(L^*) & & \mathbb{P}((L^d)^*) \arrow[u] \arrow[r] & \mathbb{P}((L^{\otimes d})^*) \arrow[u]
\end{tikzcd}
\tag{3.10}
$$

where all the maps are embeddings.

We will use the Hermitian product $H$ on $L^*$ to define a Hermitian product $\mathbf{H}$ on the vector space $\oplus_\beta (L^{\otimes d})^*$. First we recall the following. Let $H_1$, $H_2$ be Hermitian products on vector spaces $V_1$, $V_2$ respectively. Then $H_1 \oplus H_2$ and $H_1 \otimes H_2$, defined respectively by the formulas

$$
\begin{aligned}
(H_1 \oplus H_2)((v_1, v_2), (w_1, w_2)) &= H_1(v_1, w_1) + H_2(v_2, w_2) \\
(H_1 \otimes H_2)(v_1 \otimes v_2, w_1 \otimes w_2) &= H_1(v_1, w_1) H_2(v_2, w_2),
\end{aligned}
\tag{3.11}
$$

for all $v_i, w_i \in V_i$, yield well-defined Hermitian products on $V_1 \oplus V_2$ and $V_1 \otimes V_2$ respectively. Recalling also that $(L^{\otimes d})^* \cong (L^*)^{\otimes d}$, we now define

$$
\mathbf{H} := \bigoplus_{\substack{\beta=(\beta_1,\dots,\beta_r)\in\mathbb{Z}^r_{\geq 0}\\ \sum_{i=1}^r i\cdot\beta_i=d}} H^{\otimes d}
\tag{3.12}
$$

on $\oplus_\beta L^{\otimes d}$. By restriction $\mathbf{H}$ induces Hermitian products on the other three spaces in the diagram (3.9). The Hermitian product on $(L^{\otimes d})^* \cong (L^*)^{\otimes d}$ induced from the diagonal inclusion $v \mapsto (v, v, \dots, v)$ coincides with the Hermitian product $N \cdot H^{\otimes d}$ where here $N$ denotes the number of components in the direct sum $\oplus_\beta L^{\otimes d}$. We now define

$$
\mathbf{\Omega} := \frac{1}{N \cdot d} \Omega_{\mathbf{H}}
\tag{3.13}
$$

where $\Omega_{\mathbf{H}}$ is the Kähler structure on $\mathbb{P}(\oplus_\beta (L^{\otimes d})^*)$ corresponding to the Hermitian product $\mathbf{H}$.

The normalization factor in (3.13) is chosen precisely so that the following holds.

**Theorem 3.5.** *The pull-back of the form $\mathbf{\Omega}$ on $\mathbb{P}(\oplus_\beta (L^{\otimes d})^*)$ to $X$, via the maps in* (3.10)*, coincides with the original Kähler form $\omega$ with respect to the Hermitian product $H$ on $L^*$.*

*Proof.* By what was said above, the pull-back of $\mathbf{\Omega}$ to $\mathbb{P}((L^d)^*)$ is equal to $(1/N \cdot d)\Omega_{H^{\otimes d}}$, where $\Omega_{H^{\otimes d}}$ denotes the Kähler structure on $\mathbb{P}((L^d)^*)$ induced from $H^{\otimes d}$. The theorem now follows from [KaKh10, Lemma 7.9], which states that the pull-back of $\Omega_{H^{\otimes d}}$ to $X$ under the Kodaira map $\Phi_{L^d}$ is equal to $d \cdot \omega$. (Recall $\omega$ is the pullback of $\Omega_H$ to $X$ under $\Phi_L$.) Since $\Omega_H$ pulls back to $\mathbb{P}((L^d)^*)$ to equal $N \cdot \Omega_{H^{\otimes d}}$, the claim follows from (3.13). $\square$

Finally, we address the invariance of the Kähler structure $\mathbf{\Omega}$ under a torus action. Let $\mathbb{T} \cong (\mathbb{C}^*)^n$ and let $T = (S^1)^n$ denote the usual compact torus in $\mathbb{T}$. Suppose $\mathbb{T}$ acts linearly on each space $(L^i)^*$ such that the Hermitian product $H^{\otimes i}$ on $(L^i)^*$ is $T$-invariant. A natural geometric situation when this arises is when $\mathbb{T}$ acts on $X$ and $\mathcal{L}$ is a $\mathbb{T}$-linearized line bundle equipped with a $T$-invariant Hermitian form. Note that the $\mathbb{T}$-actions on the $(L^i)^*$ induce a $\mathbb{T}$-action on $\mathbf{V}_d^*$.

**Proposition 3.6.** *Suppose $T$ acts linearly on $L^i$ and suppose the $T$-action preserves $H^{\otimes d}$ with respect to the induced $T$-action on $\mathbf{V}_d^*$. Then the Hermitian product $\mathbf{H}$ on $\mathbf{V}_d^*$ is $T$-invariant. In particular, the corresponding Kähler structure $\mathbf{\Omega}$ on $\mathbb{P}(\mathbf{V}_d^*)$ is $T$-invariant.*

*Proof.* Let $T$ act on a vector space $V$ equipped with a Hermitian product $H$. Then the $T$-action preserves $H$ if and only if there exists an orthonormal basis of $V$ consisting of $T$-weight vectors. For each $i$, let $\{f_{i1}^*, \ldots, f_{in_i}^*\}$ be a basis consisting of $T$-weight vectors. By assumption, the $T$-action preserves $H^{\otimes i}$, so we may assume without loss of generality that the basis is orthonormal. Under the identification $\mathrm{Sym}((L^1)^* \otimes (L^2)^* \otimes \cdots \otimes (L^r)^*) \cong \mathbb{C}[f_{ij}; 1 \le i \le r, 1 \le j \le n_i]$ (compare (3.2) and (3.3)), the set of vectors

$$\mathbf{B} = \left\{ \prod_{i,j} (f_{ij}^*)^{\alpha_{ij}} \,\middle|\, \alpha = (\alpha_{ij}); \sum_{i=1}^{r} i \cdot \sum_{j=1}^{n_i} \alpha_{ij} = d; \alpha_{ij} \in \mathbb{Z}_{\ge 0}) \right\} \tag{3.14}$$

is a basis of $T$-weight vectors for $\mathbf{V}_d^*$. Moreover, from the definition of $\mathbf{H}$, it follows that $\mathbf{B}$ is an orthonormal basis, as desired. □

## 4. Toric degeneration

We now return to a discussion of the homogeneous coordinate ring $R$ of $X$ and the semigroup $S = S(R, v, h)$ of Definition 2.7. We retain the assumption that $v$ is a valuation with one-dimensional leaves. Moreover, from now on, we place the additional assumptions that $X$ is *smooth*, and $S$ is *finitely generated*. (The latter assumption implies that the Okounkov body $\Delta(R)$ is a rational polytope.) The semigroup algebra $\mathbb{C}[S]$ of a semigroup $S \subseteq \mathbb{N} \times \mathbb{Z}^n$ is defined to be the subalgebra of $\mathbb{C}[t, x_1, \ldots, x_n]$ spanned by the monomials $t^k x_1^{a_1} \cdots x_n^{a_n}$ for all $(k, a_1, \ldots, a_n) \in S$. Recall that $S$ is the image of the extended valuation $\tilde{v} : R \setminus \{0\} \to \mathbb{N} \times \mathbb{Z}^n$ of (2.4). In [KaKh08], the authors mentioned briefly that the homogeneous coordinate ring $R$ of $X$ can be degenerated in a flat family to the semigroup algebra $\mathbb{C}[S]$ associated to the value semigroup $S = S(R)$. In fact, Teissier proved already in 1999 that there always exists a toric degeneration from an algebra to the semigroup algebra associated to a valuation [Te99]. (More generally, the existence of a degeneration of an algebra to its associated graded with respect to a filtration is well-known and classical in commutative algebra [Ei95, Chapter 6].) In the context of Okounkov bodies, this idea was developed further, with suggestions on applications to other areas such as Schubert calculus, in [An10]. Geometrically, the existence of the algebraic degeneration means that the variety $X$ can be degenerated, in a flat family $\mathfrak{X}$, to the (not necessarily normal) projective toric variety $X_0$ associated to the semigroup $S$. The normalization of $X_0$ is then the (normal) toric variety corresponding to the polytope $\Delta(R)$.

Our construction of an integrable system uses Anderson's toric degeneration as a key ingredient, so in this section we both review and slightly expand on his exposition in [An10]. First we recall how the semigroup algebra $\mathbb{C}[S]$ of $R$ is related to the data of the extended valuation $\tilde{v}$. The link comes from a filtration of $R$ defined using $\tilde{v}$ as follows. Let

$$R_{\ge (m,u)} := \{ f \in R \,|\, \tilde{v}(f) \ge (m, u) \text{ or } f = 0 \}$$

and similarly for $R_{>(m,u)}$. By the definition of the valuation $v$ and the definition of the order (2.3) on $\mathbb{N} \times \mathbb{Z}^n$, each $R_{\geq(m,u)}$ (and $R_{>(m,u)}$) is a (finite-dimensional) vector subspace of $R$. Moreover, again by properties of valuations, for any $(m,u)$ and $(m',u')$, we have

$$R_{\geq(m,u)} \cdot R_{\geq(m',u')} \subseteq R_{\geq(m+m',u+u')}.$$

We can now define the associated graded ring $\mathrm{gr}R$ with respect to this filtration as

$$\mathrm{gr}R := \bigoplus_{(m,u)} R_{\geq(m,u)}/R_{>(m,u)}.$$

This is naturally an $S$-graded ring. In fact, since $\tilde{v}$ has one-dimensional leaves, the spaces $R_{\geq(m,u)}/R_{>(m,u)}$ have dimension 1 precisely when $(m,u) \in S$, and is 0 otherwise. Since the homogeneous elements of $\mathrm{gr}R$ are not zero divisors, one shows that $\mathrm{gr}R$ is isomorphic to the semigroup algebra $\mathbb{C}[S]$ (see [BrGu09, Remark 4.13]).

As mentioned above, there is a well-known method of degenerating an algebra $R$ to its associated graded $\mathrm{gr}R$ corresponding to a filtration of $R$ in a flat family. This is explained in more detail in [An10]. The construction depends on a choice of collection of sections $\{f_{ij}\} \in R$ which we now explain.

**Definition 4.1.** Let $R = \bigoplus_i R_i$ and $\tilde{v}$ as above. Let $r > 0$ be a positive integer and $n_i > 0$ be positive integers for each $1 \leq i \leq r$. Let $\{f_{ij}\}_{1\leq i\leq r, 1\leq j\leq n_i}$ be a finite collection of elements in $R$, where each $f_{ij}$ is homogeneous with respect to the $\mathbb{N}$-grading on $R$ (coming from the decomposition $R = \bigoplus_i R_i$). We say that this collection $\{f_{ij}\}$ form a **SAGBI basis** (or a **subductive set**) **for** $(R, \tilde{v})$ if

(a) the set of images $\{\tilde{v}(f_{ij}\}$ generate the semigroup $S = S(R) = S(R, v, h)$.

This definition generalizes the notion of SAGBI bases (Subalgebra Analogue of Gröbner Basis for Ideals) for a subalgebra of a polynomial ring, and was introduced in [Ka11] and [Ma11]. In addition, by re-indexing if necessary, we assume without loss of generality that the $\{f_{ij}\}$ also have the property that

(b) $f_{ij} \in R_i$ for all $1 \leq i \leq r, 1 \leq j \leq n_i$.

In fact, by adding additional elements if necessary, we also assume that

(c) for each $i$, the collection $\{f_{i1}, f_{i2}, \ldots, f_{in_i}\}$ is a vector space (i.e. additive) basis for $R_i$.

For the remainder of this discussion, we fix a SAGBI basis $\{f_{ij}\}_{1\leq i\leq r, 1\leq j\leq n_i}$ of $R$ satisfying the conditions (a)-(c) in Definition 4.1 above. Since the $\tilde{v}(f_{ij})$ generate the semigroup $S(R)$, using the classical and simple *subduction algorithm*, we can show that every element of $R$ can be represented as a polynomial in the $f_{ij}$.

**Proposition 4.2.** *Any $f \in R$ can be written as a polynomial in the $\{f_{ij}\}$. In particular, a SAGBI basis $\{f_{ij}\}$ of $R$ satisfying the conditions (a)-(c) of Definition 4.1 generates $R$ as an algebra.*

*Proof.* For each $f_{ij}$ in the chosen SAGBI basis, let $\tilde{v}(f_{ij}) = (i, u_{ij})$ for $u_{ij} \in \mathbb{Z}^n$. Let $f \in R$ and let $\tilde{v}(f) = (m,u)$. Since $\tilde{v}(f_{ij})$ generate $S$ as a semigroup, there exist $k_{ij} \in \mathbb{Z}_{\geq 0}$ such that $(m,u) = \sum_{ij} k_{ij}(i, u_{ij})$. In particular, $\tilde{v}(f) = \tilde{v}(\prod_{i,j} f_{ij}^{k_{ij}})$. Moreover, since $v$ (and hence $\tilde{v}$) has one-dimensional leaves, there exists a nonzero constant $\lambda \in \mathbb{C}$ such that

$$g_1 := f - \lambda \prod_{ij} f_{ij}^{k_{ij}}$$

has the property that either $\tilde{v}(f) > \tilde{v}(g_1)$ or $g_1 = 0$. If $g_1 = 0$, then we are done. Otherwise, notice that $g_1 \in \oplus_{k=1}^m R_k$. Repeating the same argument for $g_1$ in place of $f$ and continuing by induction, we obtain a sequence $g_1, g_2, \dots$ of elements of $\oplus_{k=1}^m R_k$ with $\tilde{v}(f) > \tilde{v}(g_1) > \tilde{v}(g_2) > \cdots$. Since $\oplus_{k=1}^m R_k$ is finite dimensional, by Proposition 2.4 we have that $\tilde{v}(\oplus_{k=1}^m R_k)$ is finite. Thus, as the values of $\tilde{v}(g_i)$ strictly decrease, there must exist some $s > 0$ where $g_s = 0$, as desired. □

We can now state the main result of this section [An10, Proposition 5.1].

**Theorem 4.3.** *Let $R$ be as above and assume $S = S(R)$ is finitely generated. Then there is a finitely generated, $\mathbb{N}$-graded, flat $\mathbb{C}[t]$-subalgebra $\mathcal{R} \subset R[t]$, such that:*

(a) $\mathcal{R}[t^{-1}] \cong R[t, t^{-1}]$ *as $\mathbb{C}[t, t^{-1}]$-algebras, and*

(b) $\mathcal{R}/t\mathcal{R} \cong \mathrm{gr}R$.

The geometric interpretation [An10, Corollary 5.3] of Theorem 4.3 is obtained by taking Proj with respect to the $\mathbb{N}$-grading on $\mathcal{R}$.

**Corollary 4.4.** *There is a projective flat family $\pi : \mathfrak{X} := \mathrm{Proj}\ \mathcal{R} \to \mathbb{C}$ such that:*

(a) *For any $z \neq 0$ the fiber $X_z = \pi^{-1}(z)$ is isomorphic to $X = \mathrm{Proj}\ R$. More precisely, the family over $\mathbb{C} \setminus \{0\}$, i.e. $\pi^{-1}(\mathbb{C} \setminus \{0\})$ is isomorphic to $X \times (\mathbb{C} \setminus \{0\})$.*

(b) *The special fiber $X_0 = \pi^{-1}(0)$ is isomorphic to* Proj $(\mathrm{gr}R) \cong \mathrm{Proj}\mathbb{C}[S]$ *and is equipped with an action of $(\mathbb{C}^*)^n$, where $n = \dim_{\mathbb{C}} X$. The normalization of the variety* Proj $(\mathrm{gr}R)$ *is the toric variety associated to the rational polytope $\Delta(R)$.*

*Sketch of proof of Theorem 4.3.* Let $\tilde{v}(f_{ij}) = (i, u_{ij})$. Let $\overline{f}_{ij}$ denote the image of $f_{ij}$ in the associated graded $\mathrm{gr}R$. Let $A = \mathbb{C}[x_{ij}]$ denote the polynomial algebra in the indeterminates $x_{ij}$, $1 \leq i \leq r, 1 \leq j \leq n_i$. Define an $(\mathbb{N} \times \mathbb{Z}^n)$-grading on $A$ by $\deg(x_{ij}) := (i, u_{ij})$; thus the surjective map $A \to \mathrm{gr}R$ defined by $x_{ij} \mapsto \overline{f}_{ij}$ is a map of graded rings. The kernel of this map is a homogeneous ideal $I_0$. Let $\overline{g}_1, \dots, \overline{g}_q$ be homogeneous generators for the kernel and let $\deg(\overline{g}_k) = (n_k, v_k)$. It follows that $\overline{g}_k(f_{11}, \dots, f_{rn_r})$ lies in $R_{>(n_k, v_k)}$ for each k. By the proof of Proposition 4.2 one can find elements $g_k \in \overline{g}_k + A_{<(n_k, v_k)}$ such that $g_k(f_{11}, \dots, f_{rn_r}) = 0$. The $g_k$ will not be homogeneous for the full $\mathbb{N} \times \mathbb{Z}^n$ grading of $A$, but since the $f_{ij}$ are homogeneous for the first $\mathbb{N}$ factor, the $g_k$ can be chosen to respect the $\mathbb{N}$-grading as well.

The induced map $A/\langle g_1, ..., g_q \rangle \to R$ is an isomorphism. In fact, if $I$ denotes the kernel of the map $A \to R$, then $g_k \in I$ by construction, and the initial terms $g_k$ generate the initial ideal $I_0$ of $I$ (which is the kernel of $A \to \mathrm{gr}R$). It follows that the $g_k$'s form a Gröbner basis for $I$, with respect to the term order determined by the order on $\mathbb{Z} \times \mathbb{Z}^n$, cf. [Ei95, Exercise 15.14(a)]. (Specifically, the term order is $\prod_{ij} x_{ij}^{a_{ij}} \leq \prod_{ij} x_{ij}^{b_{ij}}$ if and only if $\sum_{ij} a_{ij}(i, u_{ij}) \leq \sum_{ij} b_{ij}(i, u_{ij})$. Note that we allow ties between monomials in this notion of term order.)

The remainder of the construction and proof will only be sketched: see [An10] for details. Here we record only the points directly relevant to our construction in Section 5. Let $\mathcal{S} \subset \mathbb{Z} \times \mathbb{Z}^n$ be a finite subset consisting of all the degrees $(s, v)$ for all the monomials appearing in any of the $g_k$. Let $p : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{Z}$ be a linear map which preserves the ordering on $\mathcal{S}$ (cf. [An10, Lemma 5.2, Proof of Prop. 5.1]). Let $w_{ij} = p(i, u_{ij})$ be the degree of $x_{ij}$ under the weighting induced by $p$. From the choice of $p$ it follows that for any $k$, the initial term of $g_k$ with respect to the weighting defined by $p$ is exactly $\overline{g}_k$. Let $\ell_k = p(n_k, v_k)$, and set

$$\tilde{g}_k = \tau^{\ell_k} g_k(\tau^{-w_{11}} x_{11}, \dots, \tau^{-w_{rn_r}} x_{rn_r}) \in A[\tau],$$

where $\tau$ is an indeterminate. Consider the homomorphism $A[\tau] \to R[t]$ given by

$$x_{ij} \mapsto t^{w_{ij}} f_{ij} \quad \text{and } \tau \mapsto t. \tag{4.1}$$

Let $\mathcal{R}$ be the image of this homomorphism; this ring satisfies all the conditions of the theorem. In addition, the ring $\mathcal{R}$ can be presented explicitly as $\mathcal{R} = A[\tau]/(\tilde{g}_1, \ldots, \tilde{g}_q)$. (This is a standard way of producing a Rees ring from a Gröbner basis; see, e.g., [Ei95, Theorem 15.17]). □

We take a moment to concretely and geometrically interpret the family $\mathcal{R}$ determined by the map (4.1). Consider the map

$$\rho : X \times \mathbb{C}^* \to \mathfrak{X} \hookrightarrow W\mathbb{P} \times \mathbb{C} = W\mathbb{P}(L^*, (L^2)^*, \ldots, (L^r)^*; 1, 2, \ldots, r) \times \mathbb{C}$$

given by

(4.2)

$$(x, t) \mapsto \left( \left( (t^{w_{11}} \frac{f_{11}(x)}{h(x)}, \ldots, t^{w_{1n_1}} \frac{f_{1n_1}(x)}{h(x)}) : \cdots : (t^{w_{r1}} \frac{f_{1r}(x)}{h^r(x)}, \ldots, t^{w_{rn_r}} \frac{f_{rn_r}(x)}{h^r(x)}) \right), t \right)$$

where $h$ is a section with $h(x) \neq 0$ (see (3.4)). The image of the family $\mathfrak{X}$ sitting inside the weighted projective space $W\mathbb{P} \times \mathbb{C}$ is the closure of the image of this map. This image is cut out by the equations $\tilde{g}_k(x_{ij}, \tau) = 0$ for $k = 1, \ldots, q$. We use the notation $\pi : \mathfrak{X} \to \mathbb{C}$ to denote the projection of $\mathfrak{X} \subseteq W\mathbb{P} \times \mathbb{C}$ to the second factor, and let $X_z$ and $X_0$ denote a general fiber $\pi^{-1}(z)$ for $z \in \mathbb{C}$ and the special fiber $\pi^{-1}(0)$ respectively.

**Remark 4.5.** By construction, the image of $X_0$ in the weighted projective space is cut out by the equations $\overline{g}_k(x_{ij}) = 0$ for $1 \leq k \leq q$. The map $\mathbb{C}[x_{ij}] \to \mathbb{C}[x_{ij}]/I_0 \cong \mathrm{gr}R$ is a homomorphism of $(\mathbb{N} \times \mathbb{Z}^n)$-graded algebras and hence the embedding $X_0 \to W\mathbb{P} \times \{0\}$ is $\mathbb{T} = (\mathbb{C}^*)^n$ equivariant. Moreover, the embedding $W\mathbb{P} \hookrightarrow \mathbb{P}(\mathbf{V}_d^*)$ is $\mathbb{T}$-equivariant.

By Corollary 4.4(a) and since $X$ is assumed to be smooth, we know that the family $\mathfrak{X}$ is nonsingular away from the special fibre $X_0$. In fact, more is true; we have the following. We thank Dave Anderson for the statement and proof below. For definitions of the technical terms used in the proof see [Ha77, Theorem 8.22A].

**Theorem 4.6.** *Let $X, R, \tilde{v}$ and $S$ be as above. In particular assume that $X$ is smooth and $S$ is finitely generated. Then the family $\mathfrak{X}$ is nonsingular away from the singular locus of the special fiber $X_0$.*

*Proof.* Take a nonsingular point $p$ of $X_0$, with $(x_1, \ldots, x_n)$ a regular sequence generating the maximal ideal (for $p$ in $X_0$). Let t be the parameter on the base; then $(t, x_1, \ldots, x_n)$ is a regular sequence for the maximal ideal of p in $\mathfrak{X}$. So $\mathfrak{X}$ is R1. In fact, it is also S2, hence normal. This is essentially because the base is a nonsingular curve and the fibers are all S1 (they have no embedded points). □

## 5. Construction of an integrable system on $X$

5.1. **Integrable systems.** We begin by recalling some background. Let $(M, \omega)$ be a symplectic manifold. A smooth function $H \in C^\infty(M)$, also called a Hamiltonian, defines a vector field $\xi_H$, called the *Hamiltonian vector field* (associated to $H$) by the equality

$$\omega(\xi_H, \cdot) = dH(\cdot). \tag{5.1}$$

The Hamiltonian vector field defines a differential equation on $M$, the *Hamiltonian system* associated to $H$. The equation (5.1) also defines in turn a *Poisson bracket* on $C^\infty(M)$ by

$$\{f, g\} := \omega(\xi_g, \xi_f). \tag{5.2}$$

A function $f$ on $M$ is said to be a *first integral* of $H$ if $\{H, f\} = 0$. A Hamiltonian system $(M, \omega, H)$ on a $2n$-dimensional symplectic manifold is *completely integrable* (or *integrable*) if there exist $n = \frac{1}{2}\dim_{\mathbb{R}} M$ first integrals $H_1, H_2, \dots, H_n$ which

- are functionally independent, i.e., there exists an open dense subset $U$ of $M$ such that, for all $x \in U$, the differentials $dH_i(x)$ are linearly independent,
- pairwise Poisson-commute, i.e., $\{H_i, H_j\} = 0$ for all $i, j$.

By slight abuse of language one often refers to the collection $\{H_1, H_2, \dots, H_n\}$ as an integrable system on $(M, \omega)$.

There are many interesting and well-studied examples of integrable systems on symplectic manifolds (cf. [Au96] and references therein). However, as far as we are aware, little is known about explicit methods for constructing interesting integrable systems on a general symplectic manifold $(M, \omega)$. Moreover, in many cases, an integrable system exists only on an open dense subset of $M$ in the sense that the chosen functions are smooth only on an open dense subset. A famous example is the Gel'fand-Cetlin system on coadjoint orbits of $U(n)$ described by Guillemin and Sternberg [GuSt83]. In such cases, it is desirable to know that the chosen functions $\{H_1, H_2, \dots, H_{\frac{1}{2}\dim_{\mathbb{R}} M}\}$ in the integrable system extend continuously to all of $M$.

The above discussion motivates the following definition.

**Definition 5.1.** Let $X$ be a complex variety of dimension $n$. Let $\omega$ be a symplectic form on the smooth locus of $X$. We call a collection of real-valued functions $\{F_1, \dots, F_n\}$ on $X$ a *completely integrable system* (or an *integrable system*) on $X$ if the following are satisfied:

(a) $F_1, \dots, F_n$ are continuous on all of $X$.
(b) There exists an open dense subset $U$ contained in the smooth locus of $X$ such that $F_1, \dots, F_n$ are differentiable on $U$, and the differentials $dF_1, \dots, dF_n$ are linearly independent on $U$.
(c) $F_1, \dots, F_n$ pairwise Poisson-commute on $U$.

We call the map $\mu := (F_1, \dots, F_n) : X \to \mathbb{R}^n$ the *moment map* of the integrable system.

Returning to the setting of previous sections, let $X$ be a smooth projective variety of dimension $n$ over $\mathbb{C}$, and hence $\dim_{\mathbb{R}}(X) = 2n$. Let $\mathcal{L}$ be a very ample line bundle on $X$, $L = H^0(X, \mathcal{L})$ and $\Phi_L : X \to \mathbb{P}(L^*)$ the corresponding Kodaira embedding. We equip $X$ with the Kähler (in particular symplectic) structure induced by the Kodaira embedding via pulling back the standard Fubini-Study form on $\mathbb{P}(L^*)$ associated to a choice of Hermitian metric $H$ on $L^*$. Our main goal of this section, and of this manuscript, is to use the toric degeneration described in Section 4 to construct an integrable system $\{F_1, \dots, F_n\}$ on $X$ in the sense of Definition 5.1.

5.2. **Integrable systems on a toric variety.** As before, let $\mathbb{T} = (\mathbb{C}^*)^n$ denote an algebraic torus. Let $T = (S^1)^n$ denote the corresponding compact torus. Let $V$ be a finite-dimensional $\mathbb{T}$-module. The action of $\mathbb{T}$ on $V$ induces an action on the projective space $\mathbb{P}(V)$. Suppose $H$ is a $T$-invariant Hermitian form on $V$. Then the corresponding Fubini-Study Kähler structure $\Omega_H$ on $\mathbb{P}(V)$ is $T$-invariant. In fact, more is true: the $T$-action on $(\mathbb{P}(V), \Omega_H)$ is Hamiltonian, with moment map $\mu : \mathbb{P}(V) \to \mathrm{Lie}(T)^*$ given by

$$\langle \mu(x), \xi \rangle = \frac{i}{2} \frac{H(\xi \cdot \tilde{x}, \tilde{x})}{H(\tilde{x}, \tilde{x})} \tag{5.3}$$

where $\xi \in \mathrm{Lie}(T)$ and $\tilde{x} \in V$ represents the point $x \in \mathbb{P}(V)$.

Let $S \subset \mathbb{N} \times \mathbb{Z}^n$ be a finitely generated semigroup. We assume that the group generated by $S$ is all of $\mathbb{Z} \times \mathbb{Z}^n \cong \mathbb{Z}^{n+1}$. The semigroup algebra $\mathbb{C}[S]$ is $\mathbb{Z}^n$-graded (by the second factor of $\mathbb{N} \times \mathbb{Z}^n$) and hence the projective variety $X_S = \mathrm{Proj}(\mathbb{C}[S])$ (taken with respect to the first $\mathbb{N}$-grading) has a $\mathbb{T}$-action. From the assumption that $S$ generates the group $\mathbb{Z}^{n+1}$ it follows that $X_S$ is a (not necessarily normal) projective $\mathbb{T}$-toric variety.

Let $\{(i, u_{ij}) \mid i = 1, \ldots, r, j = 1, \ldots, n_i\}$ be a finite set of generators for $S$. The choice of generators $u_{ij} \in \mathbb{Z}^n$ gives rise to an embedding of $\mathbb{T}$ into the weighted projective space $W\mathbb{P} = W\mathbb{P}(\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_r}; 1, \ldots, r)$, defined by the monomial map

$$x \mapsto ((x^{u_{11}}, \ldots, x^{u_{1n_1}}) : \cdots : (x^{u_{r1}}, \ldots, x^{u_{rn_r}})) \tag{5.4}$$

where for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ the notation $x^a$ denotes the monomial $x_1^{a_1} \cdots x_n^{a_n}$. Here the torus $\mathbb{T}$ acts diagonally on each vector space $\mathbb{C}^{n_i}$ by the weights $\{u_{i1}, \ldots, u_{in_i}\}$. These actions of $\mathbb{T}$ induce an action of $\mathbb{T}$ on $W\mathbb{P}$ and the embedding $X_S \hookrightarrow W\mathbb{P}$ is $\mathbb{T}$-equivariant. The closure of the image of $\mathbb{T}$ in $W\mathbb{P}$ is the toric variety $X_S$. Consider the embedding of $W\mathbb{P}$ into $\mathbb{P}(\mathbf{V}_d^*)$, where $d = r!$, given in (3.3). The image of $X_S \subseteq W\mathbb{P}$ in $\mathbb{P}(\mathbf{V}_d^*)$ is the closure of the image of $\mathbb{T}$ under the monomial map $\mathbb{T} \to \mathbb{P}(\mathbf{V}_d^*)$ given by

$$x \mapsto \left( x^{u_\alpha} \;\middle|\; \alpha = (\alpha_{ij}), \sum_{i=1}^{r} \sum_{j=1}^{n_i} i\alpha_{ij} = d, \alpha_{ij} \in \mathbb{Z}_{\geq 0} \right)$$

where for each $\alpha = (\alpha_{ij})$, we define $u_\alpha := \sum_{i=1}^{r} \sum_{j=1}^{n_i} \alpha_{ij} u_{ij}$. With respect to the diagonal $\mathbb{T}$-action on $\mathbb{P}(\mathbf{V}_d^*)$ via the weights $u_\alpha$, the inclusion $X_S \hookrightarrow W\mathbb{P} \hookrightarrow \mathbb{P}(\mathbf{V}_d^*)$ is $\mathbb{T}$-equivariant. Moreover, since the integral points $(i, u_{ij})$ generate the semigroup $S$, the set

$$\left\{ (d, u_\alpha) \;\middle|\; \alpha = (\alpha_{ij}), \sum_{i=1}^{r} \sum_{j=1}^{n_i} i\alpha_{ij} = d, \alpha_{ij} \in \mathbb{Z}_{\geq 0} \right\}$$

coincides with the set $S_d$ of elements of $S$ at level $d$. Thus the convex hull of this set is the polytope $d\Delta(S)$, where $\Delta(S)$ is the convex polytope (2.5) associated to $S$.

Let $\mathbf{H}$ be a $T$-invariant Hermitian form on $\mathbf{V}_d^*$ and let $\Omega_{\mathbf{H}}$ be the corresponding Fubini-Study Kähler form on $\mathbb{P}(\mathbf{V}_d^*)$. Consider the Kähler form $\mathbf{\Omega} := (1/d)\Omega_{\mathbf{H}}$ on $\mathbb{P}(\mathbf{V}_d^*)$. Via the embedding $X_S \hookrightarrow \mathbb{P}(\mathbf{V}_d^*)$, the smooth locus of the toric variety $X_S$ inherits a Kähler (hence symplectic) structure from $\mathbb{P}(\mathbf{V}_d^*)$. Moreover, the $T$-action on this smooth locus is Hamiltonian, with moment map $\mu_S = \frac{1}{d}\mu|_{X_S}$, where $\mu$ is the map in (5.3). In particular, the image of the moment map $\mu_S$ is the polytope $(1/d)d\Delta(S) = \Delta(S)$.

Let $\{H_1, \ldots, H_n\}$ denote the components of $\mu_S : X_S \to \mathrm{Lie}(T)^* \cong \mathbb{R}^n$ with respect to some choice of basis of $\mathrm{Lie}(T)$. Since $X_S$ is a toric variety, i.e. has a dense open orbit isomorphic to $\mathbb{T}$ itself, we immediately conclude:

- The collection $\{H_1, \ldots, H_n\}$ is an integrable system on $X_S$ in the sense of Definition 5.1.
- The image of the moment map $\mu_S = (H_1, \ldots, H_n)$ is the rational polytope $\Delta(S)$.
- By construction, the Hamiltonians $H_1, \ldots, H_n$ generate a torus action on $X_S$.

5.3. **Kähler structure on the family $\mathfrak{X}$.** Returning to our previous setting, let $X$ be a projective variety with $\mathcal{L}$ a very ample line bundle, $L = H^0(X, \mathcal{L})$, and $R$ its homogeneous coordinate ring. Let $v$ be a valuation with one-dimensional leaves on $\mathbb{C}(X)$ and assume $S = S(R)$ is finitely generated. Also let $\Delta = \Delta(R)$ denote the associated Okounkov body. Since $S$ is finitely generated, $\Delta$ is a rational polytope.

In Section 4 we described a flat family $\mathfrak{X}$ such that general fibers $X_z$, $z \neq 0$, are isomorphic to $X$ and the central fiber $X_0$ is the (possibly non-normal) toric variety associated to the semigroup $S$. Recall that the family $\mathfrak{X}$ lies in $W\mathbb{P} \times \mathbb{C}$ where the weighted projective space $W\mathbb{P}$ is $W\mathbb{P}(L^*, \ldots, (L^r)^*; 1, \ldots, r)$ for some $r > 0$. By embedding $W\mathbb{P}$ in $\mathbb{P}(\mathbf{V}_d^*)$ as in Section 3.1, we henceforth think of the family $\mathfrak{X}$ as a subvariety in the smooth variety $\mathbb{P}(\mathbf{V}_d^*) \times \mathbb{C}$. We equip $\mathbb{P}(\mathbf{V}_d^*) \times \mathbb{C}$ with the product symplectic structure $\Omega \times \left(\frac{i}{2} dz \wedge d\bar{z}\right)$ where $z$ is the complex parameter on the $\mathbb{C}$ factor and so $\frac{i}{2} dz \wedge d\bar{z}$ is the standard Kähler structure on $\mathbb{C}$. We denote by $\tilde{\omega}$ (respectively $\omega_z$) the restriction of this product structure to the smooth locus of the family $\mathfrak{X}$ (respectively the fiber $X_z$). In fact, by Theorem 4.6, the family is smooth away from the singular locus of $X_0$, so each $(X_z, \omega_z)$ is a smooth Kähler manifold. When we write $\omega_0$, we mean the Kähler structure on the smooth locus of $X_0$.

Recall that the construction of the family $\mathfrak{X}$ involved a choice of elements $\{f_{ij}\}$ in $R$, satisfying properties (a)-(c) in Definition 4.1. The following lemma proves that we may also assume, without loss of generality, that in addition to the properties (a)-(c), the collection $\{f_{ij}\}$ is compatible with a choice of Hermitian metric.

**Lemma 5.2.** *Let $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ be a valuation with one-dimensional leaves. Let $V \subset \mathbb{C}(X)$ be a finite-dimensional vector subspace and let $H$ be a Hermitian product on the dual space $V^*$. Then there exists a basis $\{b_1, \ldots, b_k\}$ of $V$ such that*

(1) *the corresponding dual basis $\{b_1^*, \ldots, b_n^*\}$ is orthonormal with respect to $H$ and*
(2) *$v(b_i) \neq v(b_j)$ for all $i \neq j$, i.e., $v(V \setminus \{0\}) = \{v(b_1), \ldots, v(b_k)\}$.*

*Proof.* From Proposition 2.4 we know that $\#v(V \setminus \{0\}) = \dim_{\mathbb{C}} V$. Let $k = \dim_{\mathbb{C}} V$ and let $\{a_1 > \cdots > a_k\} = v(V \setminus \{0\})$, listed in decreasing order. For $a \in \mathbb{Z}^n$ let $V_{\geq a} := \{x \mid v(x) \geq a\} \cup \{0\}$. This is a subspace of $V$. We have a flag of subspaces

$$\{0\} \subsetneqq V_{a_1} \subsetneqq \cdots \subsetneqq V_{a_k} = V.$$

Using the Hermitian product on $V^*$, we define an identification $\phi : V^* \to V$ which thus also equips $V$ with a Hermitian product by pullback via $\phi$. In such a situation an orthonormal basis of $V$ is identified via $\phi$ with its own dual basis in $V^*$, which is also orthonormal. Thus, the desired basis can be constructed by finding an orthonormal basis of $V_{a_1}$, then inductively an orthonormal basis for $V_{a_s}$ extending the given one on $V_{a_{s-1}}$. □

From this lemma it follows that we may assume that our SAGBI basis $\{f_{ij}\}$ has the property that, for each fixed $i, 1 \leq i \leq r$, the collection $\{f_{ij}\}_{1 \leq j \leq n_i}$ is a basis for $L^i$ such that its dual basis $\{f_{i1}^*, \ldots, f_{in_i}^*\}$ is orthonormal in $(L^i)^*$, with respect to the Hermitian product on $(L^i)^*$ induced from $H^{\otimes i}$.

Now let the torus $\mathbb{T}$ act linearly on each $(L^i)^*$ such that every $f_{ij}^*$ is a $T$-weight vector with weight $u_{ij} = v(f_{ij})$. Note this corresponds to the action on $W\mathbb{P}(L^*, (L^2)^*, \ldots, (L^r)^*; 1, 2, \ldots, r)$ implicitly described in (5.4) where we have identified $(L^i)^* \cong \mathbb{C}^{n_i}$ using the basis $\{f_{i1}^*, \ldots, f_{in_i}^*\}$. From the orthogonality of the basis $\{f_{i1}^*, \ldots, f_{in_i}^*\}$ it follows that the induced action of $T$ on $(L^i)^*$ preserves the Hermitian product.

For $0 \neq z \in \mathbb{C}$ let $\rho_z$ denote the restriction of the embedding $\rho : X \times \mathbb{C}^* \to \mathfrak{X} \to W\mathbb{P} \times \mathbb{C}$ given in (4.2) to $X \times \{z\}$.

**Proposition 5.3.** (a) *Under the isomorphism $\rho_1 : X \to X_1 \subset \mathfrak{X}$, the pullback $\rho_1^* \omega_1$ of the Kähler structure $\omega_1$ on $X_1$ is equal to the original Kähler structure $\omega$ on $X$.*

(b) *The Kähler structure $\omega_0$ on the (smooth locus of) the $\mathbb{T}$-toric variety $X_0 \subset \mathbb{P}(\mathbf{V}_d^*) \times \{0\} \cong \mathbb{P}(\mathbf{V}_d^*)$ is invariant under the $T$-action.*

*Proof.* The embedding

$$\rho_1 : X \hookrightarrow W\mathbb{P} \times \{1\}$$

coincides with the embedding $\Psi : X \to W\mathbb{P}$ in diagram (3.6). This proves (a). Part (b) follows from Proposition 3.6 where $\mathbb{T}$ acts on each space $L^i$ in such a way that the $f_{ij}$ are weight vectors with weights $u_{ij}$. $\square$

Fix $0 \neq \epsilon \in \mathbb{C}$. Consider the automorphism $\Lambda_\epsilon : W\mathbb{P} \to W\mathbb{P}$ given by

$$\Lambda_\epsilon : (x_{11} : \cdots : x_{rn_r}) \mapsto (\epsilon^{w_{11}} x_{11} : \cdots : \epsilon^{w_{rn_r}} x_{rn_r})$$

where the $w_{ij}$ are as in the proof of Theorem 4.3. The automorphism $\Lambda_\epsilon$ lifts to an automorphism $\mathbf{\Lambda}_\epsilon : \mathbb{P}(\mathbf{V}_d^*) \to \mathbb{P}(\mathbf{V}_d^*)$ which in coordinates (with respect to the basis $\mathbf{B}$ in (3.14)) is given by

$$\mathbf{\Lambda}_\epsilon : (x_\alpha) \mapsto (c_\alpha x_\alpha),$$

where $c_\alpha = \prod_{i=1}^r \prod_{j=1}^{n_i} \epsilon^{\alpha_{ij} w_{ij}}$. We clearly have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_1 \subset W\mathbb{P} \times \{1\} & \longrightarrow & \mathbb{P}(\mathbf{V}_d^*) \times \{1\} \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \Lambda_\epsilon} & & \big\downarrow{\scriptstyle \mathbf{\Lambda}_\epsilon} \\ X & \longrightarrow & X_\epsilon \subset W\mathbb{P} \times \{t\} & \longrightarrow & \mathbb{P}(\mathbf{V}_d^*) \times \{\epsilon\} \end{array} \tag{5.5}$$

Let $\mathbf{\Omega}_\epsilon$ denote the pull-back $\mathbf{\Lambda}_{1/\epsilon}^*(\mathbf{\Omega})$ of the Kähler form $\mathbf{\Omega}$ under the automorphism $\mathbf{\Lambda}_{1/\epsilon} : \mathbb{P}(\mathbf{V}_d^*) \to \mathbb{P}(\mathbf{V}_d^*)$.

**Proposition 5.4.** (a) *Under the isomorphism $\rho_\epsilon : X \to X_\epsilon \subset \mathfrak{X} \hookrightarrow W\mathbb{P}$ the Kähler form $\mathbf{\Omega}_{\epsilon|X_\epsilon}$ pulls back to the form $\omega$ on $X$.*
(b) *The Kähler structure $\mathbf{\Omega}_\epsilon$ is $T$-invariant.*

*Proof.* (a) Follows from Proposition 5.3(a) and commutativity of the diagram (5.5). (b) In the basis $\mathbf{B}$ in (3.14) the automorphism $\mathbf{\Lambda}_\epsilon$ and the torus $T$ act by diagonal matrices and hence $\mathbf{\Lambda}_\epsilon$ and $T$-action on $\mathbb{P}(\mathbf{V}_d^*)$ commute. Since $\mathbf{\Omega}$ is $T$-invariant it follows that $\mathbf{\Omega}_\epsilon$ is also $T$-invariant. $\square$

5.4. **Gradient-Hamiltonian flow.** Recall that $\pi : \mathfrak{X} \to \mathbb{C}$ is the restriction to the family $\mathfrak{X}$ of the usual projection of $\mathbb{P}(\mathbf{V}_d^*) \times \mathbb{C}$ to its second factor. In particular, $\pi$ is a holomorphic function on the smooth locus $\mathfrak{X}_{smooth} \subseteq \mathfrak{X}$, which is a smooth Kähler manifold. Following Ruan [Ru99], we now define the **gradient-Hamiltonian vector field** corresponding to $\pi$ on $\mathfrak{X}_{smooth}$ as follows. Let $\nabla(\mathrm{Re}(\pi))$ denote the gradient vector field on $\mathfrak{X}_{smooth}$ associated to the real part $\mathrm{Re}(\pi)$, with respect to the Kähler metric. Since $\tilde{\omega}$ is Kähler and $\pi$ is holomorphic, the Cauchy-Riemann equations imply that $\nabla(\mathrm{Re}(\pi))$ is related to the Hamiltonian vector field $\xi_{\mathrm{Im}(\pi)}$ of the imaginary part $\mathrm{Im}(\pi)$ with respect to the Kähler (symplectic) form $\tilde{\omega}$ by

$$\nabla(\mathrm{Re}(\pi)) = -\xi_{\mathrm{Im}(\pi)}. \tag{5.6}$$

Let $Z \subset \mathfrak{X}$ denote the closed set which is the union of the singular locus of $\mathfrak{X}$ (which by Theorem 4.6 is contained in $X_0$) and the critical set of $\mathrm{Re}(\pi)$, i.e. the set on which $\nabla(\mathrm{Re}(\pi)) = 0$. The *gradient-Hamiltonian vector field* $V_\pi$, which is defined only on the open set $\mathfrak{X} \setminus Z$, is by definition

$$V_\pi := -\frac{\nabla(\mathrm{Re}(\pi))}{\|\nabla(\mathrm{Re}(\pi))\|^2}. \tag{5.7}$$

Where defined, $V_\pi$ is smooth. By definition we have

$$V_\pi(\mathrm{Re}(\pi)) = -1. \tag{5.8}$$

For $t \in \mathbb{R}_{\geq 0}$ let $\phi_t$ denote the time-$t$ flow corresponding to $V_\pi$. Since $V_\pi$ may not be complete, $\phi_t$ for a given $t$ is not necessarily defined on all of $\mathfrak{X} \setminus Z$. We record the following facts.

**Proposition 5.5.** (a) *Where defined, the flow $\phi_t$ takes $X_s \cap (\mathfrak{X} \setminus Z)$ to $X_{s-t}$. In particular, where defined, $\phi_t$ takes a point $x \in X_t$ to a point in the special fiber $X_0$.*

(b) *Where defined, the flow $\phi_t$ preserves the symplectic structures, i.e., if $x \in X_z \cap (\mathfrak{X} \setminus Z)$ is a point where $\phi_t(x)$ is defined, then $\phi_t^*(\omega_{z-t})_{\phi_t(x)} = (\omega_z)_x$.*

Recall that a smooth point $x \in \mathfrak{X}$ is called a critical point of $\pi$ if $d\pi(x) = 0$. A scalar $z \in \mathbb{C}$ is a critical value if $X_z$ contains at least one critical point. Note that by the Cauchy-Riemann relations $d\pi = 0$ if and only if $d(\mathrm{Re}(\pi)) = 0$, which in turn holds if and only if $\nabla(\mathrm{Re}(\pi)) = 0$. The next lemma is a corollary of the Sard and Tarski theorems.

**Lemma 5.6.** *The set of critical values of $\pi$ is a finite set. Thus there exists $\epsilon > 0$ such that there is no critical value in the interval $(0, \epsilon]$.*

*Proof.* The set $\mathrm{Crit}(\pi) \subset \mathfrak{X}$ of critical points of $\pi$ is a complex semi-algebraic set. By the complex version of Tarski's theorem, the image of $\mathrm{Crit}(\pi)$, that is, the set of critical values, is also a semi-algebraic set in $\mathbb{C}$. Thus it is either finite or it has finite complement. On the other hand, by Sard's theorem, the set of critical values must be of measure zero in $\mathbb{C}$. Thus it must be finite. $\square$

Next we recall the following fundamental theorem on the solutions of ordinary differential equations (see for example [HuLi05, Theorem 1.7.1]):

**Theorem 5.7.** *Let $V$ be a continuously differentiable vector field defined in an open subset $U$ of a Euclidean space. Then for any $x_0 \in U$, the flow $\phi_t$ of $V$ is defined at $x_0$ for $0 \leq t < b$ where we have the following possibilities:*

1. *$b = \infty$ or*
2. *$b < \infty$ and $\phi_t(x_0)$ is unbounded as $t \to b^-$, or*
3. *$b < \infty$ and $\phi_t(x_0)$ approaches the boundary of $U$ as $t \to b^-$.*

Let $U_0 \subset X_0$ denote the open set $X_0 \setminus (X_0 \cap Z)$. Then the gradient-Hamiltonian vector field is defined at all $x_0 \in U_0$. In the next section we prove that the flatness of the family $\mathfrak{X}$ implies that $U_0$ is in fact dense in $X_0$.

**Lemma 5.8.** *Let $\epsilon > 0$ be such that $\epsilon < \epsilon'$ for an $\epsilon' > 0$ satisfying the condition in Lemma 5.6, i.e., there are no critical values in $(0, \epsilon'] \subset \mathbb{C}$. Then the flow $\phi_{-\epsilon}$ is defined for all $x_0 \in U_0$.*

*Proof.* Consider the gradient-Hamiltonian vector field $V_\pi$, defined on $\mathfrak{X} \setminus Z$. Let $X_{[0,\epsilon')} := \pi^{-1}([0, \epsilon')) \subset \mathfrak{X}$. We can consider the vector field $V_\pi$ restricted to the submanifold $X_{[0,\epsilon')} \setminus (X_{[0,\epsilon')} \cap Z)$. Let $x_0 \in U_0$ and suppose for a contradiction that $\phi_{-t}(x_0)$ is defined only for $0 \leq t < b$ where $b \leq \epsilon < \epsilon'$. The set $X_{[0,\epsilon']} := \pi^{-1}([0, \epsilon'])$ is compact, so from Theorem 5.7 we conclude that, as $t \to b^-$, the values $\phi_{-t}(x_0)$ approach the boundary of $X_{[0,\epsilon')} \setminus (X_{[0,\epsilon')} \cap Z)$, which is contained in $X_0 \cup X_{\epsilon'}$. On the other hand, by Proposition 5.5, $\phi_{-t}(x_0) \in X_t$ and hence, as $t \to b^-$, $\phi_{-t}(x_0)$ approaches $X_b$, which is disjoint from $X_0 \cup X_{\epsilon'}$ since $b < \epsilon'$. This is a contradiction. Thus $\phi_{-t}(x_0)$ must be defined for $t = \epsilon$ for any $x_0 \in U_0$, as desired. $\square$

5.5. **Flatness and the critical set of $\pi$ on the special fiber $X_0$.** In this section we show that the gradient-Hamiltonian flow (of $f = \mathrm{Re}(\pi)$) is defined on a non-empty Zariski-open subset of the special fiber $X_0$. In the notation of the previous section, this will imply that $U_0$ is open and dense in $X_0$. To prove the claim, it is enough to show that the differential $d\pi|_{X_0}$ is non-zero on a non-empty Zariski-open subset.

In fact, we will prove a more general statement. Let $\pi : \mathfrak{Y} \to \mathbb{C}$, $\mathfrak{Y} \subset \mathbb{P}^N \times \mathbb{C}$, be a flat family of irreducible varieties, where $\pi$ denotes the restriction to $\mathfrak{Y}$ of the projection to the second factor of $\mathbb{P}^N \times \mathbb{C} \to \mathbb{C}$. Let $Y_z$ denote the fiber $\pi^{-1}(z)$, which by assumption is an irreducible variety. We have the following.

**Theorem 5.9.** *Let $z \in \mathbb{C}$ and suppose $\mathfrak{Y}$ is smooth at general points of $Y_z$. Then the differential $d\pi$ is not identically equal to $0$ on the smooth locus of $Y_z$.*

Recall that for a projective subvariety $Y \subset \mathbb{P}^N$, the degree of $Y$ can be computed as the number of intersection points of $Y$ with a general plane of complementary dimension, i.e., if $\dim_{\mathbb{C}}(Y) = n$ and $L$ is a general plane in $\mathbb{P}^N$ of codimension $n$ then $\deg(Y) = \#(Y \cap L)$. In terms of the intersection product of algebraic cycles in $\mathbb{P}^N$, the degree of $Y$ is the degree of the 0-cycle $Y \cdot L$. We also recall that the leading coefficient of the Hilbert polynomial $H_Y(k)$ of $Y \subset \mathbb{P}^N$ is $\frac{\deg(Y)}{n!}$ where $\dim_{\mathbb{C}}(Y) = n$. The following is well-known (cf. [Ha77, Chapter 3, Theorem 9.9]).

**Theorem 5.10.** *Suppose $\mathfrak{Y} \subset \mathbb{P}^N \times \mathbb{C}$ is a flat family. Then the Hilbert polynomials $H_{Y_z}(k)$ are equal for all $z \in \mathbb{C}$. In particular, the degrees of $Y_z$ as subvarieties of the projective space $\mathbb{P}^N \cong \mathbb{P}^N \times \{z\}$ are equal, for all $z \in \mathbb{C}$.*

With Theorem 5.10 in hand, we prove Theorem 5.9 by looking at the intersection product of cycles in the non-singular variety $\mathbb{P}^N \times \mathbb{C}$. First we recall some basic facts about the intersection product, referring to [Fu98] for more details. Let $V$ and $W$ be closed subvarieties of $\mathbb{P}^N \times \mathbb{C}$, of pure dimensions $k$ and $\ell$ respectively. Suppose $V$ and $W$ intersect properly, i.e. each irreducible component $Z$ of $V \cap W$ has dimension $k + \ell - (N+1)$. Then the *intersection product* $V \cdot W$ can be written as

$$V \cdot W = \sum_Z i(Z, V \cdot W) Z, \tag{5.9}$$

where $Z$ runs over the irreducible components of $V \cap W$, and $i(Z, V \cdot W) > 0$ is the *intersection multiplicity* of $V \cdot W$ at $Z$ [Fu98, Section 8.2]. We have the following [Fu98, Proposition 8.2 and Remark 8.2].

**Proposition 5.11.** *Let $V, W$ be as above and let $Z$ be an irreducible component of $V \cap W$. If $Z$ is a single point $\{x\}$ then the intersection multiplicity $i(Z, V \cdot W)$ is equal to $1$ if and only if $V$ and $W$ are non-singular at $x$ and meet transversely at $x$. If $\dim_{\mathbb{C}}(Z) > 0$, then $i(Z, V \cdot W)$ is equal to $1$ if and only if $V$ and $W$ are generically non-singular along $Z$ and generically meet transversely along $Z$.*

We are now in a position to prove our claim.

*Proof of Theorem 5.9.* For $z \in \mathbb{C}$ let $H_z$ denote the non-singular irreducible hypersurface $\mathbb{P}^N \times \{z\}$. Then by definition $Y_z = \mathfrak{Y} \cap H_z$. Since $\mathfrak{Y}$ is a flat family of irreducible varieties, the $Y_z$ are irreducible with the same dimension $n$. Thus $\mathfrak{Y}$ and $H_z$ intersect properly for all $z$. If $x \in Y_z$ is a smooth point of both $Y_z$ and $\mathfrak{Y}$, then $d\pi(x)$ is non-zero if and only if $T_x\mathfrak{Y}$ is not contained in $T_x H_z$, i.e. if and only if $\mathfrak{Y}$ and $H_z$ are transverse at $x$. Hence it

now suffices to show that the intersection multiplicity $i(Y_z, \mathfrak{Y} \cdot H_z)$ is equal to 1 for all $z$. By (5.9) for any $z \in \mathbb{C}$ we can write

$$\mathfrak{Y} \cdot H_z = i(Y_z, \mathfrak{Y} \cdot H_z) Y_z.$$

Let $E$ be a hyperplane in $\mathbb{P}^N$ of codimension $n$. Then $E \times \mathbb{C}$ is a non-singular irreducible subvariety of $\mathbb{P}^N \times \mathbb{C}$ of codimension $n$. Consider the intersection product

$$\mathfrak{Y} \cdot H_z \cdot (E \times \mathbb{C}) = i(Y_z, \mathfrak{Y} \cdot H_z) Y_z \cdot (E \times \mathbb{C}).$$

Let $d$ denote the intersection number of $\mathfrak{Y}$, $H_z$ and $E \times \mathbb{C}$, i.e., the degree of the 0-cycle $\mathfrak{Y} \cdot H_z \cdot (E \times \mathbb{C})$. Notice that for different $z$ (respectively different $E$) the $H_z$ (respectively $E \times \mathbb{C}$) are rationally equivalent (in $\mathbb{P}^N \times \mathbb{C}$). Thus we see that the intersection number $d$ is independent of the choice of $t$ and $E$. Moreover, since $E \times \mathbb{C}$ is transverse to $H_z = \mathbb{P}^N \times \{z\}$, the degree of the 0-cycle $Y_z \cdot (E \times \mathbb{C})$ is equal to the degree of the variety $Y_z$ (as a subvariety of $\mathbb{P}^N \times \mathbb{C}$); hence

$$d = i(Y_z, \mathfrak{Y} \cdot H_z) \deg(Y_z).$$

Since $\mathfrak{Y}$ is a flat family, we know from Theorem 5.10 that the degree of $Y_z$ is independent of $z$. From this we conclude that the intersection multiplicity $i(Y_z, \mathfrak{Y} \cdot H_z)$ is independent of $z \in \mathbb{C}$. Finally, by Bertini's theorem, there exists $z$ such that $H_z$ is generically transverse to $\mathfrak{Y}$. By Proposition 5.11 we conclude that for all $z$ the intersection multiplicity $i(Y_z, \mathfrak{Y} \cdot H_z)$ is equal to 1, as desired. □

The following is immediate.

**Corollary 5.12.** *Let $U_0$ denote the open subset $X_0 \setminus (X_0 \cap Z)$. Then $U_0$ is open and dense in $X_0$.*

We will also need the following.

**Corollary 5.13.** *Let $\epsilon > 0$ be as in Lemma 5.8. Let $U_0$ denote the open subset $X_0 \setminus (X_0 \cap Z)$. Then the image $U_\epsilon = \phi_{-\epsilon}(U_0)$ of $U_0$ under the flow $\phi_{-\epsilon}$ is an open dense subset of $X_\epsilon$.*

*Proof.* The flow $\phi_{-\epsilon}$ is a local diffeomorphism. Since $\phi_{-\epsilon}$ sends $U_0$ to $X_\epsilon$, the image $U_\epsilon := \phi_{-\epsilon}(U_0)$ is an open subset of $X_\epsilon$. It remains to show that $U_\epsilon$ is dense in $X_\epsilon$. To this end, recall that by Proposition 5.5 we know $\phi_{-\epsilon}^*(\omega_\epsilon) = \omega_0$. Thus

$$\int_{X_0} \omega_0^n = \int_{U_0} \omega_0^n = \int_{U_0} (\phi_{-\epsilon}^*(\omega_\epsilon))^n = \int_{U_\epsilon} \omega_\epsilon^n \tag{5.10}$$

where the first equality uses Corollary 5.12. On the other hand, the symplectic volumes $\int_{X_0} \omega_0^n$ and $\int_{X_\epsilon} \omega_\epsilon^n$ are equal (up to a constant) to the degrees of $X_0$ and $X_\epsilon$ regarded as subvarieties of $\mathbb{P}(\mathbf{V}_d^*)$ [Mu76, Section 5C]. Since the family $\mathfrak{X}$ is flat, the degrees of $X_0$ and $X_\epsilon$ are equal by Theorem 5.10. Thus

$$\int_{X_0} \omega_0^n = \int_{X_\epsilon} \omega_\epsilon^n. \tag{5.11}$$

Equations (5.10) and (5.11) together imply that $\int_{U_\epsilon} \omega_\epsilon^n = \int_{X_\epsilon} \omega_\epsilon^n$. Thus $X_\epsilon \setminus U_\epsilon$ has empty interior and $U_\epsilon$ is dense in $X_\epsilon$ as desired. □

5.6. **Continuity of the flow.** We return to the setting of Section 5.4. In this section we prove that for a small enough parameter $\varepsilon > 0$, the flow $\phi_\varepsilon$, which is a priori not defined at an arbitrary point of $X_\varepsilon$, can be extended to a continuous function on all of $X_\varepsilon$.

**Theorem 5.14.** *Let $\epsilon > 0$ satisfy the condition of Lemma 5.8. Then there exists an $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \epsilon$ such that the flow $\phi_\varepsilon$, a priori defined only on an open dense subset $U_\varepsilon \subseteq X_\varepsilon$, extends to a continuous function defined on all of $X_\varepsilon$, with image in $X_0$.*

**Remark 5.15.** In general, the flow with respect to the gradient vector field of a smooth function $f$ on a Riemannian manifold $X$ does not give rise to a continuous function between the level sets of $f$. The reason that we can continuously extend our flow $\phi_\varepsilon$ is that we work in an algebraic setting.

The proof of Theorem 5.14 is based on the famous "Łojasiewicz gradient inequality" for analytic functions (cf. [Lo84], also [KuMoPa00, pages 763 and 765]).

**Theorem 5.16.** *Let $f$ be a real-valued analytic function defined on some open subset $W \subset \mathbb{R}^m$. Then for any $x \in W$ there exists an open neighborhood $U_x$ of $x$ and constants $c_x > 0$ and $0 < \alpha_x < 1$ such that for all $y \in U_x$:*

$$||\nabla f(y)|| \geq c_x |f(y) - f(x)|^{\alpha_x}. \tag{5.12}$$

In fact, since $X_0$ may not be smooth, we will need the following generalization of Theorem 5.16 due to Kurdyka and Parusinski [Ku94, Proposition 1].

**Theorem 5.17.** *Let $X$ be a (possibly singular) algebraic subset of $\mathbb{R}^m$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a semi-algebraic function. Then for any $x \in X$ (not necessarily a smooth point) there is an open neighborhood $U_x \subset X$ (in the usual topology) and constants $c_x > 0$ and $0 < \alpha_x < 1$ such that for any smooth point $y \in U_x$ we have:*

$$||\nabla f(y)|| \geq c_x |f(y) - f(x)|^{\alpha_x}, \tag{5.13}$$

*where $\nabla f$ denotes the gradient of $f|_X$ with respect to the induced metric on the smooth locus of $X$.*

**Remark 5.18.** Since any Riemannian metric on a relatively compact subset of $\mathbb{R}^n$ (i.e. a subset with compact closure) is equivalent to the Euclidean metric, the inequalities (5.12) and (5.13) hold for an arbitrary Riemannian metric on $\mathbb{R}^m$ with the same exponent $\alpha_x$ and possibly different constant $c_x$.

**Remark 5.19.** In [Ku94] the more general case of subanalytic functions and sets is addressed. For our purposes, the algebraic/semi-algebraic case stated above suffices.

*Proof of Theorem 5.14.* Proposition 5.5 shows that, where defined, the flow $\phi_t$ takes a fiber $X_s$ to $X_{s-t}$. Since the argument is essentially unchanged by doing so and to simplify the notation in what follows, we will now assume that $\varepsilon = 1$. In particular we assume that, for any $0 < t < 1$, the fiber $X_t$ contains no critical points of $\pi$. Thus for any point $x \in X_1$ and any $0 < t < 1$, the flow $\phi_t(x)$ is defined.

For $x \in X_0$, let $U_x$, $c_x > 0$ and $0 < \alpha_x < 1$ be as in Theorem 5.17. Since $X_0$ is compact, it can be covered with finitely many such open sets $U_{x_1}, \ldots, U_{x_s}$. Let

$$U := \left( \bigcup_{i=1}^{s} U_{x_i} \right) \cap \{ y \mid |f(y)| < 1 \},$$

$$c := \min\{c_{x_1}, \ldots, c_{x_s}\},$$

$$\alpha := \max\{\alpha_1, \dots, \alpha_s\}.$$

Clearly $X_0 \subset U$. From Theorem 5.17 it also follows that for any point $y \in U \setminus X_0$ we have $||\nabla f(y)|| \geq c|f(y)|^{\alpha}$, and hence

$$||\nabla f(y)||^{-1} \leq (1/c)|f(y)|^{-\alpha}. \tag{5.14}$$

Next we claim that there exists $\rho > 0$ such that $X_t \subset U$ for all $0 < t < \rho$. To see this, suppose for a contradiction that there exists a sequence $(x_i)_{i\in\mathbb{N}}$, such that for all $i$ we have $x_i \notin U$, $0 < \pi(x_i) < 1$, and $\lim_{i\to\infty} \pi(x_i) = 0$. From the compactness of $\mathbb{P}(\mathbf{V}_d^*) \times [0, 1]$ it follows that the sequence $(x_i)$ has a limit point $x \in \mathfrak{X}$. Since $x_i \notin U$ for all $i$ and $U$ is open, we know $x \notin U$. On the other hand, by continuity $\pi(x) = 0$ and hence $x \in X_0 \subset U$, contradiction. Now let $\rho > 0$ be as above. Then for any $1 - \rho < t < 1$ and for any $y \in X_1$ we have

$$\phi_t(y) \in X_{1-t} \subset U.$$

Let $x \in X_1$. Then for any $t_0, t_1$ with $1 - \rho < t_0 < t_1 < 1$, we have

$$\begin{aligned} \|\phi_{t_1}(x) - \phi_{t_0}(x)\| &= \left\| \int_{t_0}^{t_1} \frac{d}{dt}\phi_t(x)dt \right\| \\ &= \left\| \int_{t_0}^{t_1} V(\phi_t(x))dt \right\| \\ &\leq \int_{t_0}^{t_1} \|V(\phi_t(x))\|\, dt \end{aligned}$$

In addition, by (5.14) we have

$$||\phi_{t_1}(x) - \phi_{t_0}(x)|| \leq \int_{t_0}^{t_1} (1/c)|f(\phi_t(x))|^{-\alpha} dt$$

and since $x \in X_1$, by Proposition 5.5 we have

$$f(\phi_t(x)) = 1 - t.$$

Putting this together we obtain

$$\begin{aligned} \|\phi_{t_1}(x) - \phi_{t_0}(x)\| &\leq \frac{1}{c}\int_{t_0}^{t_1} (1-t)^{-\alpha} dt \\ &\leq \frac{1}{c(1-\alpha)}((1-t_1)^{1-\alpha} - (1-t_0)^{1-\alpha}). \end{aligned}$$

Since $1-\alpha > 0$ by assumption, we know $\lim_{t\to 1}(1-t)^{1-\alpha} = 0$. We have just proved that, for any $\varepsilon' > 0$, there exists $\delta > 0$ such that for any $x \in X_1$ and any $t_0, t_1$ with $1-\delta < t_0 < t_1 < 1$, we have

$$||\phi_{t_1}(x) - \phi_{t_0}(x)|| \leq \varepsilon'.$$

It follows that for any $x \in X_1$, the limit $\lim_{t\to 1} \phi_t(x)$ exists. Let us denote this limit by $\phi(x)$. By continuity of the flow, if the original flow $\phi_1$ was already defined at $x$, then $\phi_1(x) = \phi(x)$. Thus the map $\phi$ is an extension of $\phi_1$ to all of $X_1$, and by construction takes $X_1$ to $X_0$.

To finish the proof of Theorem 5.14 it remains to show that $\phi : X_1 \to X_0$ is continuous. Let $x \in X_1$. We need to show that for any $\varepsilon'' > 0$ there exists $\delta > 0$ such that if $y \in X_1$ and $||y - x|| < \delta$ then $||\phi(y) - \phi(x)|| < \varepsilon''$. By the above, we can choose $t \in \mathbb{R}$ with $0 < t < 1$ such that for any $y \in X_1$ we have

$$||\phi_t(y) - \phi(y)|| < \varepsilon''/3.$$

Since the map $y \mapsto \phi_t(y)$ is continuous, we can find $\delta > 0$ such that for any $y \in X_1$, if $||x - y|| < \delta$ then $||\phi_t(x) - \phi_t(y)|| < \varepsilon/3$. Now, if $||x - y|| < \delta$, we have:

$$\begin{aligned} ||\phi(x) - \phi(y)|| &\leq ||\phi(x) - \phi_t(x)|| + ||\phi_t(x) - \phi_t(y)|| + ||\phi_t(y) - \phi(y)|| \\ &\leq \varepsilon''/3 + \varepsilon''/3 + \varepsilon''/3 \\ &\leq \varepsilon'' \end{aligned}$$

as desired. □

5.7. **Construction of an integrable system on $X$.** With these preliminaries in place, we can now prove the main result of this paper.

**Theorem 5.20.** *Let $X$ be a smooth projective variety equipped with $\mathcal{L}$ a very ample line bundle and let $R$ denote its homogeneous coordinate ring. Let $v$ be a valuation with one-dimensional leaves on $\mathbb{C}(X)$ and assume the associated semigroup $S = S(R)$ is finitely generated. Let $\Delta = \Delta(R)$ denote the associated Okounkov body. Then there exists a (completely) integrable system $\{F_1, \ldots, F_n\}$ on $X$, in the sense of Definition 5.1, such that the image of the moment map $\mu := (F_1, \ldots, F_n) : X \to \mathbb{R}^n$ is the polytope $\Delta$.*

*Proof.* Let $\{H_1, \ldots, H_n\}$ be an integrable system on the projective toric variety $X_0$ with moment image $\Delta$ as in Section 5.2. Let $\varepsilon > 0$ be as found in Theorem 5.14. Consider the flow of the gradient-Hamiltonian vector field associated to $\mathrm{Re}(\pi)$. We saw in Section 5.4 that the flow $\phi_{-\varepsilon}$ is a diffeomorphism from an open dense subset $U_0 \subset X_0$ to an open dense subset $U_\varepsilon \subset X_\varepsilon$.

By Proposition 5.4, by replacing the Kähler structure $\mathbf{\Omega}$ on $\mathbb{P}(\mathbf{V}_d^*)$ with $\mathbf{\Omega}_\varepsilon$, without loss of generality we may assume that $\varepsilon = 1$. Then the pull-back of Kähler structure $\omega_1$ on $X_1$ under the isomorphism $\rho_1 : X \to X_1 \subset \mathfrak{X}$ coincides with $\omega$. Moreover, Theorem 5.14 shows that the flow $\phi_1 : U_1 \to U_0$ extends to a continuous function $\phi : X_1 \to X_0$. Hence we have a sequence of maps

$$X \xrightarrow{\rho_1} X_1 \xrightarrow{\phi} X_0$$

where $\rho_1$ is a symplectomorphism and $\phi$ is continuous and its restriction to an open dense subset $U_0$ is a symplectomorphism. Now define a collection of functions $\{F_1, \ldots, F_n\}$ on $X$ by

$$F_i = (\phi \circ \rho_1)^* H_i, \text{ for all } 1 \leq i \leq n$$

where the $\{H_1, \ldots, H_n\}$ is the integrable system on the toric variety $X_0$ found in Section 5.2. We claim that the $\{F_1, \ldots, F_n\}$ form a completely integrable system on $X$ in the sense of Definition 5.1. First, by construction the functions $\{H_1, \ldots, H_n\}$ pairwise Poisson-commute with respect to $\omega_0$ and their differentials are linearly independent on a dense open set $U_0$ in $X_0$. Since $\phi \circ \rho_1$ is a symplectomorphism from $\rho_1^{-1}(U_1)$ to $U_0$, it follows that the $\{F_1, \ldots, F_n\}$ also pairwise Poisson-commute with respect to $\omega$ on the dense open subset $\rho_1^{-1}(U_1)$ of $X$. The continuity of the $F_i$ follows from the continuity of $\phi \circ \rho_1$ on all of $X$ and the continuity of the $H_i$. Finally, since $X$ is compact, the image of the continuous map $\phi \circ \rho_1 : X \to X_0$ is closed in $X_0$. On the other hand, this image contains the dense open set $U_0$ and hence $\phi \circ \rho_1$ is surjective. It follows that the image of $(F_1, \ldots, F_n) : X_1 \to \mathbb{R}^n$ is the same as the image of $(H_1, \ldots, H_n) : X_0 \to \mathbb{R}^n$, which is the polytope $\Delta = \Delta(R)$, as desired. □

5.8. **Generating torus action.** In this section we show that the flat family in Section 4 can be constructed so that the integrable system in Section 5.7 generates a torus action on the dense open subset $U$. More precisely:

**Theorem 5.21.** *The flat family $\mathfrak{X}$ can be constructed in such a way that the set of points $(x, 0) \in X_0$ at which the gradient-Hamiltonian flow is defined is $T$-invariant, and hence the integrable system on $X$ generates a torus action on a dense open subset.*

Theorem 5.21 is a corollary of the following proposition:

**Proposition 5.22.** *The family $\mathfrak{X}$ can be constructed in such a way that the function $\pi : \mathfrak{X} \to \mathbb{C}$ has no critical points on (the smooth locus) of the zero fiber $X_0$.*

Recall from Section 4 that the family $\mathfrak{X} \subset W\mathbb{P} \times \mathbb{C}$ is given as the zero set of the system of equations

$$\tilde{g}_k(x_{ij}, t) := t^{\ell_k} g_k(t^{-w_{ij}} x_{ij}) = 0, \quad k = 1, \ldots, s.$$

Let $\tilde{\mathfrak{X}} \subset (L^*) \times \cdots \times (L^r)^* \times \mathbb{C}$ be the zero set of the system of equations $\tilde{g}_k(x_{ij}, t) = 0$, $k = 1, \ldots, s$. Since each $g_k$ is homogeneous with respect to the grading given by $\deg(x_{ij}) = i$, the variety $\tilde{\mathfrak{X}}$ is invariant under the action of $\mathbb{C}^*$ on $(L^*) \times \cdots \times (L^r)^*$ given by $t \cdot (x_{ij}) := (t^i x_{ij})$. The image of $\tilde{\mathfrak{X}}$ in $W\mathbb{P} := ((L^*) \times \cdots \times (L^r)^*)//(\mathbb{C}^*)$ is the family $\mathfrak{X}$. Let $(x, 0) \in X_0 \subset \mathfrak{X}$ be a smooth point and let $(\tilde{x}, 0) \in \tilde{\mathfrak{X}}$ be a smooth point lying above $(x, 0)$. The tangent space $T_{(\tilde{x},0)}\tilde{\mathfrak{X}}$ is the null space of the matrix

$$\begin{pmatrix} \frac{\partial \tilde{g}_1}{\partial x_{11}} & \cdots & \frac{\partial \tilde{g}_1}{\partial x_{rn_r}} & \frac{\partial \tilde{g}_1}{\partial t} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \frac{\partial \tilde{g}_s}{\partial x_{11}} & \cdots & \frac{\partial \tilde{g}_s}{\partial x_{rn_r}} & \frac{\partial \tilde{g}_s}{\partial t} \end{pmatrix} \tag{5.15}$$

at $(\tilde{x}, 0)$. One observes that, for each $k$, the partial derivative $(\partial/\partial t)\tilde{g}_k$ evaluated at $t = 0$ is the sum of terms in $g_k$ that contain $t$ with exponent 1. In other words, if

$$\tilde{g}_k(x, t) = \sum_{q=0}^{\ell_k} h_{k,q}(x) t^q, \tag{5.16}$$

then $(\partial \tilde{g}_k/\partial t)_{|t=0} = h_{k,1}(x)$.

**Lemma 5.23.** *The linear projection $p : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{Z}$ in the construction of the degenerating family $\mathfrak{X}$ can be chosen so that for each $k$ we have $h_{k,1} = 0$.*

*Proof.* As above for $1 \leq k \leq s$ let

$$\tilde{g}_k(x, t) = t^{\ell_k} g_k(t^{-w_{ij}} x_{ij}) = \sum_{q=0}^{\ell_k} h_{k,q}(x) t^q.$$

Let us write

$$g_k(x) = \sum_{\alpha=(\alpha_{ij})} c_\alpha \prod_{ij} x_{ij}^{\alpha_{ij}}.$$

Then from definition of $\tilde{g}$ one sees that for any $q$:

$$h_{k,q}(x) = \sum_{\{\alpha | q = \ell_k - w \cdot \alpha\}} c_\alpha \prod_{ij} x_{ij}^{\alpha_{ij}}, \tag{5.17}$$

where $w = (w_{ij})$ and $w \cdot \alpha$ denotes $\sum_{ij} w_{ij}\alpha_{ij}$. Recall that $\mathbb{C}[x_{ij}]$ has an $\mathbb{N} \times \mathbb{N}^n$-grading where $\deg(x_{ij}) = (i, u_{ij})$. With respect to this grading each monomial $x^\alpha$ has degree $\sum_{ij} \alpha_{ij}(i, u_{ij})$. Let $\mathcal{S}$ be as in the proof of Theorem 4.3 and also let $p : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{Z}$ be a linear function which preserves order in $\mathcal{S}$. Replacing $p$ with $2p$ if necessary, we can assume that $p(s, v) > 1$ for all $(0, 0) \neq (s, v) \in \mathcal{S}$. This means that for any two monomials $x^\alpha, x^\beta$ appearing in some $g_k$ we have $p(\sum_{ij} \alpha_{ij}(i, u_{ij})) - p(\sum_{ij} \beta_{ij}(i, u_{ij})) \neq 1$. Note that, $\ell_k$ (which is the degree in $t$ of the polynomial $g_k$) is exactly the maximum of $p(\sum_{ij} \alpha_{ij}(i, u_{ij}))$ for all monomials $x^\alpha$ appearing in $g_k$. Also since by definition for each $i, j$, $w_{ij} = p(i, u_{ij})$, we see that $w \cdot \alpha$ equals $p(\sum_{ij} \alpha_{ij}(i, u_{ij}))$. Thus in particular, $\ell_k - w \cdot \alpha \neq 1$, for any $\alpha$ appearing in $g_k$. From equation (5.17) we now conclude that $h_{k,1} = 0$ as required. $\square$

From Lemma 5.23 we readily obtain a proof of Proposition 5.22.

*Proof of Proposition 5.22.* Take $(x, 0) \in X_0$. The tangent space to $W\mathbb{P} \times \mathbb{C}$ at $(x, 0)$ can be written as $(L^*) \times \cdots \times (L^r)^*$ quotient by the 1-dimensional subspace which is the tangent space to the $(\mathbb{C}^*)$-orbit of $(\tilde{x}, 0)$. From Lemma 5.23 it follows that for any smooth point $(\tilde{x}, 0) \in \tilde{\mathfrak{X}}$, the tangent space $T_{(\tilde{x},0)}\tilde{\mathfrak{X}}$ contains the vector $(0, 1)$ because the last column of the matrix in (5.15) is 0. Since $\mathbb{C}^*$ acts trivially on the $\mathbb{C}$ component, it follows that $T_{(x,0)}\mathfrak{X}$ should also contain the vector $(0, 1) \in T_x W\mathbb{P} \times \mathbb{C}$. On the other hand, if $(x, 0) \in \mathfrak{X}$ is a smooth point which is also a critical point of $\pi : \mathfrak{X} \to \mathbb{C}$ then $T_{(x,0)}\mathfrak{X}$ should be vertical i.e. the vector $(0, 1)$ is orthogonal to $T_{(x,0)}\mathfrak{X}$. The contradiction shows that (the smooth locus of) the fiber $X_0$ contains no critical points of $\pi$, as desired. $\square$

*Proof of Theorem 5.21.* The gradient-Hamiltonian flow is undefined exactly at the singular points or the critical points of $f$. According to Proposition 5.22 the family can be constructed so that there are no critical points of $f$ on the zero fiber $X_0$. The statement follows by noting that the set of singular points of $X_0$ is $T$-invariant. $\square$

## 6. GIT and symplectic quotients

The main goal of this section is to show that the construction of both the flat family (the toric degeneration) $\mathfrak{X}$ and the integrable system in Sections 4 and 5, respectively, can be made compatible with the presence of a torus action. To accomplish this goal, however, the toric degeneration $\mathfrak{X}$ must be constructed using appropriate choices of the auxiliary data of the valuation $\tilde{v}$ and the SAGBI basis $\{f_{ij}\}$. We begin with a discussion of the case of toric varieties, which both sets the stage and motivates what follows.

6.1. **GIT quotients of toric varieties.** We return to the setting of Section 5.2. Let $\mathbb{T} = (\mathbb{C}^*)^n$ denote, as before, the $n$-dimensional algebraic torus with lattice of characters $\mathbb{Z}^n$. Now suppose $\mathbb{H} \subset \mathbb{T}$ is an algebraic subtorus of dimension $m$ with lattice of characters $M$. The inclusion $\mathbb{H} \subset \mathbb{T}$ gives a surjective homomorphism $\lambda : \mathbb{Z}^n \to M$, which in turn extends to a linear map $\lambda_\mathbb{R} : \mathbb{R}^n \to M_\mathbb{R} = M \otimes \mathbb{R}$. Let $\tilde{\lambda} : \mathbb{Z}^{n+1} \to M$ be a homomorphism such that the restriction of $\tilde{\lambda}$ to $\{0\} \times \mathbb{Z}^n$ coincides with $\lambda$. Fix $(1, a_0) \in S$ and define $\lambda_0 := \tilde{\lambda}(1, a_0)$. Then for any $(k, a) \in \mathbb{Z}^{n+1}$, the fact that $\tilde{\lambda}$ is a homomorphism implies

$$\tilde{\lambda}(k, a) = \lambda(a - ka_0) + k\lambda_0$$

so the lift $\tilde{\lambda}$ is determined by $\lambda$ and by its value at $(1, a_0)$. We will return to this point later. The lifted homomorphism $\tilde{\lambda}$ gives an $M$-grading on the semigroup algebra $\mathbb{C}[S]$ where for $(k, a) \in S$, the monomial $t^k x^a$ has degree $\tilde{\lambda}(k, a)$. We can uniquely extend $\tilde{\lambda}$ to a linear function $\tilde{\lambda}_\mathbb{R} : \mathbb{R}^{n+1} \to M_\mathbb{R}$. Let $\Lambda'_\mathbb{R} \subset \mathbb{R}^{n+1}$ denote the kernel of $\tilde{\lambda}_\mathbb{R}$ and let $S' = S \cap \Lambda'_\mathbb{R}$

be the subsemigroup of $S$ obtained by intersecting with the subspace $\Lambda'_{\mathbb{R}}$. The kernel of the linear map $\lambda_{\mathbb{R}}$ has dimension $n-m$ and hence the subspace $\Lambda'_{\mathbb{R}}$ has dimension $n-m+1$.

**Lemma 6.1.** *Let $S$ and $S'$ be as above, and let $\Delta = \Delta(S)$, $\Delta' = \Delta(S')$ be the convex bodies associated to the semigroups $S$ and $S'$ respectively. Then:*

(a) *$S'$ is a finitely generated semigroup.*
(b) $(\{1\} \times \Delta') = (\{1\} \times \Delta) \cap \Lambda'_{\mathbb{R}}$.

*Proof.* For part (a), recall that by assumption, $S$ is a finitely generated semigroup. Let $\{(k_i, a_i)\}_{1 \leq i \leq \ell}$ be a set of generators for $S$. Consider the homomorphism $\theta : \mathbb{R}^\ell \to \mathbb{R}^{n+1}$ defined by

$$\theta : (x_1, \ldots, x_\ell) \mapsto \sum_{i=1}^{\ell} x_i (k_i, a_i).$$

The inverse image $\theta^{-1}(\Lambda'_{\mathbb{R}})$ is a vector subspace of $\mathbb{R}^\ell$ and $C := \theta^{-1}(\Lambda'_{\mathbb{R}}) \cap \mathbb{R}^\ell_{\geq 0}$ is a convex rational polyhedral cone in $\mathbb{R}^\ell$. By Gordan's lemma (see e.g. [CoLiSc11, Proposition 2.17]), the set $C \cap \mathbb{Z}^\ell$ of integral points in this cone is a finitely generated semigroup. The image under $\Theta$ of $C \cap \mathbb{Z}^\ell$ is precisely $S'$, so the claim follows. Part (b) follows from the definition (2.5) of the convex body associated to a semigroup. $\square$

The $M$-grading on $\mathbb{C}[S]$ gives rise to an $\mathbb{H}$-action on the variety $X_S := \mathrm{Proj}\ \mathbb{C}[S]$. By construction, the $\mathbb{H}$-invariant subalgebra $\mathbb{C}[S]^{\mathbb{H}}$, which is the homogeneous-degree-0 part of $\mathbb{C}[S]$ with respect to the $M$-grading, is the semigroup algebra $\mathbb{C}[S']$. Let $X'_S := \mathrm{Proj}\ \mathbb{C}[S']$. The algebra $\mathbb{C}[S']$ is $\mathbb{N} \times \mathbb{Z}^n$ graded and hence the variety $X'_S$ has a $\mathbb{T}$-action. By definition the variety $X'_S$ is the GIT quotient of the variety $X_S$ with respect to the $\mathbb{H}$-action. In fact, it is a toric variety for the action of a torus $\mathbb{T}'$ which is a quotient of $\mathbb{T}$ (by a subgroup containing, but possibly larger than, $\mathbb{H}$). The complex dimension of the variety $X'_S$ (and the torus $\mathbb{T}'$) is equal to the real dimension of the polytope $\Delta'$.

We equip $X_S$ with a symplectic structure as in Section 5.2. Let $\mu_H : X_S \to \mathrm{Lie}(H)^* = M_{\mathbb{R}}$, $\mu_T : X_S \to \mathrm{Lie}(T)^* \cong \mathbb{R}^n$ denote the moment maps for the actions of the compact tori $H \subset \mathbb{H}$ and $T \subset \mathbb{T}$, respectively, on $X_S$. Recall that by the Kempf-Ness theorem the GIT quotient $X'_S$ can also be realized as the symplectic quotient $\mu_H^{-1}(0)/H$ of $X_S$ at 0, provided that 0 is a regular value of $\mu_H$. The GIT quotient $X'_S$ inherits a symplectic (in fact Kähler) structure coinciding with the quotient symplectic structure on $X'_S$ [MuFoKi94, Section 8.3]. Let $\mu_{T'} : X'_S \to \mathrm{Lie}(T')^*$ denote the moment map of $X'_S$ regarded as a Hamiltonian $T'$-space. Then the following diagram

(6.1)
$$\begin{tikzcd}
 & \Delta_H & \subset \ \mathrm{Lie}(H)^* \\
X_S \arrow[ur, "\mu_H"] \arrow[r, "\mu_T"] & \Delta \arrow[u, "\lambda_{\mathbb{R}}"'] & \subset \ \mathrm{Lie}(T)^* \\
\mu_H^{-1}(0) \arrow[u, hook, "i"] \arrow[d, "p"] & & \\
X'_S \arrow[r, "\mu_{T'}"] & \Delta' \arrow[uu, hook] & \subset \ \mathrm{Lie}(T')^*
\end{tikzcd}$$

commutes, where $i : \mu_H^{-1}(0) \hookrightarrow X_S$ is the inclusion map, and $p : \mu_H^{-1}(0) \to \mu_H^{-1}(0)/H = X'_S$ is the symplectic quotient map. Moreover, the image of $\mathrm{Lie}(T')^*$ in $\mathrm{Lie}(T)^*$ lies in the kernel of the linear map $\lambda_{\mathbb{R}}$.

6.2. **Invariant valuations.** We now consider a more general situation. As in Section 6.1, let $\mathbb{H} \cong (\mathbb{C}^*)^m$ denote an $m$-dimensional algebraic torus with character group $M \cong \mathbb{Z}^m$. Let $X$ be a projective $\mathbb{H}$-variety of dimension $n$. Note in particular that we do not assume $X$ is an $\mathbb{H}$-toric variety, and it may be that $m$ is strictly less than $n$. The $\mathbb{H}$-action on $X$ induces an $\mathbb{H}$-action on the field of rational functions $\mathbb{C}(X)$.

Suppose $\mathcal{L}$ is an $\mathbb{H}$-linearized very ample line bundle on $X$. Then $L = H^0(X, \mathcal{L})$ is a finite-dimensional $\mathbb{H}$-module and the homogeneous coordinate ring $R = R(L)$ is a graded $\mathbb{H}$-algebra. We define the **weight semigroup** of $R$ to be

$$S_{\mathbb{H}}(R) := \{(k, \lambda) \mid \text{ there exists } f \in R_k \text{ with } t \cdot f = t^{\lambda} f \text{ for all } t \in \mathbb{H}\} \subset \mathbb{N} \times M.$$

If $R$ is finitely generated as an algebra, then $S_{\mathbb{H}}(R)$ is a finitely generated semigroup. Hence the convex body $\Delta_{\mathbb{H}} := \Delta(S_{\mathbb{H}}(R))$ associated via (2.5) to the semigroup $S_{\mathbb{H}}(R)$ is a rational polytope. Following [Br86] and [KaKh-CB], we call $\Delta_{\mathbb{H}}$ the *moment polytope* of the $\mathbb{H}$-algebra $R$. The terminology is motivated from the well-known fact that the polytope $\Delta_{\mathbb{H}}$ coincides with the (closure of the) image of the moment map of $X_{smooth}$ regarded as a Hamiltonian $H$-space with respect to an $H$-invariant Kähler structure on $\mathbb{P}(L^*)$ (here $H \subset \mathbb{H}$ denotes the maximal compact torus in $\mathbb{H}$).

We now fix an $\mathbb{H}$-invariant valuation $v : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ with one-dimensional leaves. Such a valuation always exists (see [Ok96, KaKh-CB]). Let $h \neq 0$ in $L$ be a $\mathbb{T}$-weight vector of weight $\lambda_0$. Following the method in Section 2.2, we now define $\tilde{v} : R \setminus \{0\} \to \mathbb{N} \times \mathbb{Z}^n$ by

$$\tilde{v}(f) = (k, v(f/h^k)) \tag{6.2}$$

for $f \in R_k \setminus \{0\}$. Also as before, $\tilde{v}$ has one-dimensional leaves. We prove below that it is also $\mathbb{H}$-invariant.

**Lemma 6.2.** *The valuation $\tilde{v}$ in* (6.2) *is $\mathbb{H}$-invariant.*

*Proof.* It suffices to show $\tilde{v}(f) = \tilde{v}(t \cdot f)$ for any $t \in \mathbb{H}$ and any $f \in R_k$. We have

$$\begin{aligned} v\left(\frac{t \cdot f}{h^k}\right) &= v\left(\frac{t \cdot f}{t \cdot h^k} \cdot \frac{t \cdot h^k}{h^k}\right) \\ &= v\left(\frac{t \cdot f}{t \cdot h^k}\right) + v\left(\frac{t \cdot h^k}{h^k}\right) \\ &= v\left(\frac{f}{h^k}\right) + v\left(t^{k\lambda_0}\right) \end{aligned}$$

where the second equality follows from properties of valuations and the third equality follows from the $\mathbb{H}$-invariance of $v$ and the fact that $g$ is an $\mathbb{H}$-weight vector of weight $\lambda_0$. But $t^{k\lambda_0}$ is a non-zero constant function on $X$ and hence $v(t^{k\lambda_0}) = 0$. The claim follows. □

Let $S = S(R) \subset \mathbb{N} \times \mathbb{Z}^n$ be the semigroup associated to the algebra $R$ and the valuation $\tilde{v}$. As before, we assume that $S(R)$ is finitely generated and that, as a group, $S$ generates all of $\mathbb{Z}^{n+1} \cong \mathbb{Z} \times \mathbb{Z}^n$. We record the following [Ok96, KaKh-CB].

**Lemma 6.3.** *Let $(k, a) \in S$.*

(a) *There exists an $\mathbb{H}$-weight vector $f \in R_k$ with $\tilde{v}(f) = (k, a)$.*
(b) *Let $\lambda$ be the $\mathbb{H}$-weight of the vector $f$ found in part (a). The association $(k, a) \mapsto \lambda$ gives a well-defined function which is a semigroup homomorphism $S \to M$.*

Since $S$ generates the group $\mathbb{Z}^{n+1}$, the semigroup homomorphism constructed in Lemma 6.3(b) extends uniquely to a group homomorphism

$$\tilde{\lambda} : \mathbb{Z}^{n+1} \to M.$$

Let $\lambda = \tilde{\lambda}|_{\{0\}\times\mathbb{Z}^n}$ denote the restriction of $\tilde{\lambda}$ to $\{0\} \times \mathbb{Z}^n \cong \mathbb{Z}^n$. As in Section 6.1 we can recover $\tilde{\lambda}$ from its restriction $\lambda$ as follows. Notice that since $\tilde{v}(h) = (1, 0)$, we have $\tilde{\lambda}(1, 0) = \lambda_0$. Then for any $(k, a) \in S$ we have $\tilde{\lambda}(k, a) = \lambda(a) + k\lambda_0$. Also as in Section 6.1, we extend $\tilde{\lambda}$ to a linear map $\tilde{\lambda}_{\mathbb{R}} : \mathbb{R}^n \to M_{\mathbb{R}}$. Let $\Lambda'_{\mathbb{R}}$ denote the kernel of $\tilde{\lambda}_{\mathbb{R}}$, and let $S'$ be the semigroup $S \cap \Lambda'_{\mathbb{R}}$. Now consider the subalgebra $R' = R^{\mathbb{H}}$, i.e., $R'$ is the 0-graded part of $R$ with respect to the $M$-grading.

**Lemma 6.4.** *The semigroup $S' := S \cap \Lambda'_{\mathbb{R}}$ coincides with the semigroup $S(R')$ associated to the algebra $R'$ and the valuation $\tilde{v}$. In particular, $S(R')$ is finitely generated.*

*Proof.* Let $(k, a) \in S(R')$. This means there exists $f \in R'_k$ with $\tilde{v}(f) = (k, a)$. Since $f \in R' = R^{\mathbb{H}}$ we have $\tilde{\lambda}(k, a) = 0$, i.e., $(k, a) \in S'$. Conversely, suppose $(k, a) \in S'$. By Lemma 6.3(a) there exists an $\mathbb{H}$-weight vector $f \in R_k$ with $\tilde{v}(f) = (k, a)$. Since $(k, a) \in \Lambda'_{\mathbb{R}} = \ker(\tilde{\lambda}_{\mathbb{R}})$, Lemma 6.3(b) implies $f$ is of $\mathbb{H}$-weight 0. Thus $f \in R'$, and $(k, a) \in S(R')$ as required. The last claim of the lemma follows from Lemma 6.1(a). □

6.3. **Integrable systems compatible with GIT and symplectic quotients.** We retain the notation of Section 6.2. In particular, $\mathbb{H}$ denotes the torus which acts on $X$ (and hence $R$). Since the valuation $\tilde{v}$ on $R$ is $\mathbb{H}$-invariant by Lemma 6.2, the $M$-grading on $R$ is compatible with $\tilde{v}$ in the sense that for $k > 0$ and $u \in \mathbb{Z}^n$, the spaces $(R_k)_{\geq u}$ and $(R_k)_{>u}$ are $M$-graded. We record the following [An10, Proposition 5.18].

Recall that $\mathcal{R}$ is the family of algebras in Theorem 4.3 which degenerates the coordinate ring $R$ to the coordinate ring of a semigroup algebra $\mathbb{C}[S]$.

**Proposition 6.5.** *The $M$-grading on $R$ can be lifted to an $M$-grading on the family $\mathcal{R}$.*

*Sketch of proof.* We use notation as in the proof of Theorem 4.3. Without loss of generality we may assume that the elements $f_{ij} \in R$ are homogeneous with respect to the $M$-grading. We additionally extend the $M$-grading to $R[t]$ by defining $\deg(t) = 0$. Then the $\mathbb{C}[t]$-algebra $\mathcal{R}$ generated by $t$ and the $\tilde{f}_{ij} = t^{w_{ij}} f_{ij}$ is an $(\mathbb{N} \times M \times \mathbb{N})$-graded subalgebra of $R[t]$. In addition, we can equip $\mathbb{C}[x_{ij}, \tau]$ with an $M$-grading by defining $\deg(\tau) = 0$ and $\deg(x_{ij}) = \deg(f_{ij}) = \tilde{\lambda}(i, u_{ij}) \in M$ where $\tilde{v}(f_{ij}) = (i, u_{ij})$. Then the map $\mathbb{C}[x_{ij}, \tau] \to \mathcal{R}$ preserves the $M$-gradings. □

Taking Proj we obtain the following.

**Corollary 6.6.** (a) *The $\mathbb{H}$-action on $X$ lifts to an $\mathbb{H}$-action on the family $\mathfrak{X}$.*

(b) *The family $\mathfrak{X}$ is an $\mathbb{H}$-invariant subvariety of $W\mathbb{P} \times \mathbb{C}$, where $\mathbb{H}$ acts on $W\mathbb{P}$ via the natural action of $\mathbb{H}$ on $L^*, (L^2)^*, \ldots, (L^r)^*$, and $\mathbb{H}$ acts on $\mathbb{C}$ trivially.*

As mentioned above, in order to make the constructions of the toric degeneration and the integrable system compatible with the $\mathbb{H}$-action, we must choose the SAGBI basis $\{f_{ij}\}$ appropriately. More specifically, we assume the following.

1. Each $f_{ij}$ is homogeneous with respect to the $M$-grading.
2. A subset of the collection $\{f_{ij}\}$ forms a SAGBI basis for $R'$. More precisely, For each $i$, $1 \leq i \leq r$, there exists $n'_i \leq n_i$ such that the $\{f_{ij}\}_{1\leq j\leq n'_i}$ are homogeneous of degree 0 with respect to the $M$-grading, and the collection $\{\tilde{v}(f_{ij})\}_{1\leq i\leq r, 1\leq j\leq n'_i}$ generate the semigroup $S(R')$.

Let $\mathcal{R}$ and $\mathcal{R}'$ denote the degenerating families corresponding to $R$ and $R'$, respectively, constructed as in Theorem 4.3, using the collections $\{f_{ij}\}_{1\le i\le r, 1\le j\le n_i}$ and $\{f_{ij}\}_{1\le i\le r, 1\le j\le n'_i}$ respectively. The following is immediate.

**Lemma 6.7.** *The algebra $\mathcal{R}'$ is the degree-0 part of $\mathcal{R}$, i.e. $\mathcal{R}' = \mathcal{R}^{\mathbb{H}}$, with respect to the $M$-grading on $\mathcal{R}$ from Proposition 6.5.*

Taking Proj again we obtain the following.

**Corollary 6.8.** *There exist flat families $\pi : \mathfrak{X} \to \mathbb{C}$ and $\pi' : \mathfrak{X}' \to \mathbb{C}$ and a morphism $p : \mathfrak{X} \to \mathfrak{X}'$ such that:*

(a) *The family $\mathfrak{X}$ is a $\mathbb{H}$-variety and the projection $\pi$ is $\mathbb{H}$-invariant.*
(b) *The family $\mathfrak{X}'$ is the GIT quotient $\mathfrak{X}//\mathbb{H}$ of $\mathfrak{X}$ by $\mathbb{H}$.*
(c) *Let $\mathfrak{X}^{ss} \subset \mathfrak{X}$ denote the set of semistable points of $\mathfrak{X}$ with respect to the $\mathbb{H}$-action and let $p : \mathfrak{X}^{ss} \to \mathfrak{X}' = \mathfrak{X}^{ss}/\mathbb{H}$ be the quotient map. Then the diagram*

$$\begin{tikzcd} \mathfrak{X}^{ss} \arrow[rr, "p"] \arrow[dr, "\pi"'] & & \mathfrak{X}' \arrow[dl, "\pi'"] \\ & \mathbb{C} & \end{tikzcd} \tag{6.3}$$

*commutes.*

(vi) *The general fibers of $\mathfrak{X}$ and $\mathfrak{X}'$ are $X$ and $X' = X//\mathbb{H}$, respectively, and the special fibers are $X_0$ and $X'_0 = X_0//\mathbb{H}$, respectively.*

Let $H$ denote the compact torus of $\mathbb{H}$. As in Section 5, let $\tilde{\omega}$ denote the Kähler structure on $\mathfrak{X}$ which restricts to the Kähler structure $\omega$ on $X$. Recall that if the Kähler structure $\Omega$ on $\mathbb{P}(L^*)$ is $H$-invariant then the Kähler structure $\mathbf{\Omega}$ on $\mathbb{P}(\mathbf{V}_d^*) \times \mathbb{C}$ constructed in Theorem 3.5 is $H$-invariant, and hence $\tilde{\omega}$ on $\mathfrak{X}$ is also $H$-invariant by Proposition 3.6.

Let $\mu_H : \mathfrak{X} \to \mathrm{Lie}(H)^*$ denote the moment map of the $H$-action on the family $\mathfrak{X}$ with respect to $\tilde{\omega}$. Moreover, we assume that $\mu$ is proper on the set of smooth points on $\mathfrak{X}$, 0 is a regular value for $\mu_H$, and $H$ acts freely on $\mu^{-1}(0)$. By the Kempf-Ness theorem, $\mathfrak{X}'$ can then also be identified with the symplectic quotient $\mu_H^{-1}(0)/H$.

The Kähler structure on the quotient $\mathfrak{X}' \cong \mu_H^{-1}(0)/H$ can be described explicitly as follows. Let $(\tilde{\omega}, \tilde{g})$ denote the Kähler (symplectic) form and the corresponding Riemannian metric on $\mathfrak{X}$, and let $(\tilde{\omega}', \tilde{g}')$ denote the induced Kähler structure on $\mathfrak{X}'$. The form $\tilde{\omega}'$ is defined by the relation

$$p^*\tilde{\omega}' = \iota^*\tilde{\omega} \tag{6.4}$$

where $p : \mu_H^{-}(0) \to \mu_H^{-1}(0)/H \cong \mathfrak{X}'$ is the quotient map and $\iota : \mu_H^{-1}(0) \hookrightarrow \mathfrak{X}$ denotes the inclusion. The quotient Riemannian metric $\tilde{g}'$ is defined by the formula

$$\tilde{g}_x(v, w) = \tilde{g}'_{p(x)}(dp_x(v), dp_x(w)) \tag{6.5}$$

where $v$ and $w$ are required to lie in the orthogonal complement to the tangent space $T_x(H\cdot x)$ of the $H$-orbit $H \cdot x$ through $x$.

Before proving the main technical proposition leading to our main Theorem 6.11, we recall a simple fact about moment maps.

**Lemma 6.9.** *Let $K$ be a compact Lie group. Suppose $(M, \omega)$ is a Hamiltonian $K$-space with moment map $\mu_K : M \to \mathrm{Lie}(K)^*$. Let $f$ be a smooth function on $M$ which is $K$-invariant. Then the Hamiltonian flow of $f$ preserves $\mu_K$, i.e., $\mu_K$ is constant along the Hamiltonian vector field of $f$.*

Let $\phi_t$ and $\phi'_t$ denote the gradient-Hamiltonian flows on $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. Note that an immediate corollary of the above lemma is that the level set $\mu_H^{-1}(0)$ is invariant under the gradient-Hamiltonian flow $\phi_t$, since $\pi$ (hence $\mathrm{Re}(\pi)$) is $H$-invariant. Thus it makes sense to talk about the gradient-Hamiltonian flow restricted to $\mu_H^{-1}(0)$. The following technical proposition states that, in our setting, the gradient-Hamiltonian flow is compatible with taking GIT (symplectic) quotient.

**Proposition 6.10.** *The quotient morphism $p : \mu_H^{-1}(0) \to \mathfrak{X}' \cong \mu_H^{-1}(0)/H$ commutes with the gradient-Hamiltonian flows. More precisely, for any $x \in \mu_H^{-1}(0)$ and any $t > 0$ such that both $\phi_t(x)$ and $\phi'_t(p(x))$ are defined, we have*

$$p(\phi_t(x)) = \phi'_t(p(x)).$$

*Proof.* It is enough to show that, whenever defined we have

$$dp_x(V_x) = V'_{p(x)} \tag{6.6}$$

for $x \in \mu_H^{-1}(0)$. Here $V$ and $V'$ denote the gradient-Hamiltonian vector fields of $\mathrm{Re}(\pi)$ and $\mathrm{Re}(\pi')$, respectively. Recall that by (5.6) we have

$$\begin{aligned} V_x &= \xi_h(x)/\|\xi_h(x)\| \\ V'_{x'} &= \xi_{h'}(x')/\|\xi_{h'}(x')\| \end{aligned}$$

for $x \in \mathfrak{X}, x' \in \mathfrak{X}'$, where $h = \mathrm{Im}(\pi)$ and $h' = \mathrm{Im}(\pi')$ respectively, and $\|\cdot\|$ denotes the norm with respect to the metrics $\tilde{g}$ and $\tilde{g}'$ respectively. Let $x \in \mu_H^{-1}(0) \subset \mathfrak{X}^{ss}$. By (6.4) we have $p^*(\tilde{\omega}'_{p(x)}) = \tilde{\omega}|_{\mu_H^{-1}(0)}$. Since $h = h' \circ p$ on $\mu_H^{-1}(0)$ by Corollary 6.8(c) we see that

$$dp_x(\xi_h) = \xi_{h'}(p(x)). \tag{6.7}$$

On the other hand, from the fact that $\pi$ (and hence $f = \mathrm{Re}(\pi)$) is $H$-invariant, we also know that the gradient $\nabla f(x)$ is orthogonal to the orbit $T_x(H \cdot x)$ for any $x \in \mu_H^{-1}(0)$. Recalling that $V_x$ is also equal to $\nabla f(x)/\|\nabla f(x)\|$, applying (6.5) to $v = w = \xi_h(x)$ we can conclude

$$||\xi_h(x)|| = ||\xi_{h'}(p(x))||. \tag{6.8}$$

The result follows. $\square$

Now let $\mu : X \to \mathbb{R}^n$ and $\mu' : X' \to \mathbb{R}^{n-m}$ be the integrable systems constructed in Theorem 5.20 corresponding to the families $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. The following is now immediate from the preceding discussion.

**Theorem 6.11.** *The following diagram is commutative:*

$$\begin{tikzcd}
 & \Delta_H & \subset & \mathrm{Lie}(H)^* \\
X \arrow[ur, "\mu_H"] \arrow[r, "\mu"] & \Delta \arrow[u, "\lambda_{\mathbb{R}}"'] & \subset & \mathbb{R}^n \\
\mu_H^{-1}(0) \arrow[u, hook, "i"] \arrow[d, "p"'] & & & \\
X' \arrow[r, "\mu'"] & \Delta' \arrow[uu, hook] & \subset & \mathbb{R}^{n-m}
\end{tikzcd} \tag{6.9}$$

## 7. Examples

In previous sections we used the notation $\mathbb{H}$ to denote an algebraic torus acting on the variety $X$ where $\dim_{\mathbb{C}}\mathbb{H} = m$ is possibly less than $n = \dim_{\mathbb{C}}X$, and we reserved the notation $\mathbb{T}$ to denote the algebraic torus whose dimension is precisely equal to $\dim_{\mathbb{C}}X$. In the discussion below, we deviate from this notation and use the notation $\mathbb{T}$ (as is standard in the literature) for the torus which acts on $X$, even when $\dim_{\mathbb{C}}\mathbb{T}$ is strictly less than $\dim_{\mathbb{C}}X$.

7.1. **Elliptic curves.** Let $X$ be an elliptic curve, and $v$ be the valuation on $\mathbb{C}(X)$ associated to a point $p \in X$. Consider the line bundle $\mathcal{L} = \mathcal{O}_X(3p)$ and $L = H^0(X, \mathcal{O}_X(3p))$ giving the cubic embedding of $X$ in $\mathbb{CP}^2$. The semigroup $S(R(L)) \subset \mathbb{N} \times \mathbb{Z}$ is generated by $(1, 0)$, $(1, 1)$, $(1, 3)$ and hence finitely generated [An10, Example 5.7]. The curve $X$ degenerates to a cuspidal cubic curve. The Okounkov body $\Delta(R(L))$ is the line segment $[0, 3]$. Theorem 5.20 gives a function $F : X \to \mathbb{R}$ which is continuous on all of $X$, differentiable on a dense open subset, with image precisely $[0, 3]$.

7.2. **Flag varieties.** Let $G$ be a connected complex reductive algebraic group , $B$ a Borel subgroup and $\mathbb{T}$ its maximal torus. We denote the weight lattice, i.e. the character lattice of $\mathbb{T}$, by $\Lambda$. Then $\Lambda^+$ (respectively $\Lambda^+_{\mathbb{R}}$) are the semigroup of dominant weights (respectively positive Weyl chamber) corresponding to the choice of $B$. We also fix a maximal compact subgroup $K$ compatible with the choice of $B$ and $\mathbb{T}$ so that $T = K \cap \mathbb{T}$ is a maximal torus of $K$.

Let $G/B$ denote the complete flag variety of $G$. Given a regular dominant weight $\lambda$, i.e. a weight in the interior of the positive Weyl chamber, the variety $G/B$ embeds in the projective space $\mathbb{P}(V_\lambda)$ as the $G$-orbit of a highest weight vector. Here $V_\lambda$ is the irreducible $G$-module with highest weight $\lambda$. More generally, let $\lambda$ be a dominant weight (possibly on the boundary of the positive Weyl chamber). Then the $G$-orbit of a highest weight vector $v_\lambda$ in the projective space $\mathbb{P}(V_\lambda)$ is a partial flag variety $X_\lambda = G/P_\lambda$ where $P_\lambda$ is the $G$-stabilizer of $v_\lambda$ in $\mathbb{P}(V_\lambda)$. Let $\mathcal{L}_\lambda$ be the restriction of the line bundle $\mathcal{O}(1)$ on the projective space $\mathbb{P}(V_\lambda)$ to $X_\lambda$. By the Borel-Weil-Bott theorem the space of sections $H^0(X_\lambda, \mathcal{L}_\lambda)$ is isomorphic to $V_\lambda^*$ as a $G$-module.

Let $N = \dim_{\mathbb{C}}(G/B)$. The so-called *string polytope* is a rational polytope in $\mathbb{R}^N$ such that the number of integral points in the polytope is equal to $\dim_{\mathbb{C}}(V_\lambda)$. In fact, more is true: the integral points in a string polytope parameterize the so-called *crystal basis* for the $G$-module $V_\lambda$ (see [Li98]). The construction of the string polytope depends on the choice of a *reduced word decomposition* $\underline{w}_0 = (s_{i_1}, s_{i_2}, \dots, s_{i_N})$ for the longest element $w_0$ in the Weyl group $W$ of $(G, T)$, where here the $s_i$ denote the simple reflections corresponding to the simple roots $\alpha_i$. Thus, the polytope associated to a dominant weight $\lambda$ and a reduced decomposition $\underline{w}_0$ is often denoted as $\Delta_{\underline{w}_0}(\lambda)$ [BeZe01, Li98].

**Remark 7.1.** The well-known Gel'fand-Cetlin polytopes [GeCe50] corresponding to irreducible representations of $GL(n, \mathbb{C})$ are special cases of the string polytopes. More precisely, let $G = \mathrm{GL}(n, \mathbb{C})$. Here the Weyl group is $W = S_n$. Choose the reduced word decomposition

$$w_0 = (s_1)(s_2 s_1)(s_3 s_2 s_1) \cdots (s_{n-1} \cdots s_1)$$

for the longest element $w_0 \in S_n$, where $s_i$ denotes the simple transposition exchanging $i$ and $i + 1$. Then $\Delta_{\underline{w}_0}(\lambda)$ can be identified (after a linear change of coordinates) with the Gel'fand-Cetlin polytope corresponding to $\lambda$. Similarly, for $G = \mathrm{Sp}(2n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$, for analogous well-chosen reduced decompositions of the corresponding longest elements of the

Weyl group, we can recover the corresponding Gel'fand-Cetlin polytopes as string polytopes [Li98].

In fact, given a reduced decomposition $\underline{w}_0$, there is a rational polyhedral cone $\mathcal{C}_{\underline{w}_0}$ in $\Lambda_{\mathbb{R}}^+ \times \mathbb{R}^N$ such that each string polytope $\Delta_{\underline{w}_0}(\lambda)$ is the slice of the cone $\mathcal{C}_{\underline{w}_0}$ at $\lambda$, i.e., $\Delta_{\underline{w}_0}(\lambda) = \mathcal{C}_{\underline{w}_0} \cap \pi^{-1}(\lambda)$ where $\pi : \Lambda_{\mathbb{R}}^+ \times \mathbb{R}^N \to \Lambda_{\mathbb{R}}^+$ is the projection on the first factor [Li98]. We note the following.

(1) We can define the polytope $\Delta_{\underline{w}_0}(\lambda) = \mathcal{C}_{\underline{w}_0} \cap \pi^{-1}(\lambda)$ for any $\lambda \in \Lambda_{\mathbb{R}}^+$.
(2) The fact that $\mathcal{C}_{\underline{w}_0}$ is a convex cone implies that $\Delta_{\underline{w}_0}(k\lambda) = k\Delta_{\underline{w}_0}(\lambda)$ for any $k > 0$. Moreover, for $\lambda_1,\ \lambda_2 \in \Lambda_{\mathbb{R}}^+$ we have $\Delta_{\underline{w}_0}(\lambda_1) + \Delta_{\underline{w}_0}(\lambda_2) \subset \Delta_{\underline{w}_0}(\lambda_1 + \lambda_2)$.
(3) One also proves that the map:

$$\pi_\lambda : (t_1, \ldots, t_N) \mapsto -\lambda + t_1\alpha_{i_1} + \cdots + t_N\alpha_{i_N}, \tag{7.1}$$

projects the string polytope $\Delta_{\underline{w}_0}(\lambda)$ onto the polytope $P(\lambda)$ which is the convex hull of the Weyl group orbit of $\lambda$.

In [Ca02] Caldero constructs a flat deformation of the flag variety $X_\lambda$ to the toric variety $X_{\lambda,w_0}$ corresponding to the string polytope $\Delta_{\underline{w}_0}(\lambda)$. The key ingredient in his construction is a multiplicativity property of the (dual) canonical basis with respect to the string parametrization. In [Ka11] it is shown that Anderson's toric degeneration recounted in Section 4 is a generalization of Caldero's construction.

Fix a $K$-invariant Hermitian metric on $\mathbb{P}(V_\lambda)$. Let $\mu_T$ denote the moment map for the Hamiltonian action of the compact torus $T \subset K$ on $X_\lambda \cong G/P_\lambda$. Theorem 5.20 applied to $(X_\lambda, \mathcal{L}_\lambda)$ implies the following:

**Corollary 7.2.** *Let $n = \dim_{\mathbb{C}}(X_\lambda)$. There exists a (completely) integrable system $\mu_\lambda = (F_1, \ldots, F_n) : X_\lambda \to \mathbb{R}^n$ such that*

(a) *the image of $\mu_\lambda$ is the string polytope $\Delta_{\underline{w}_0}(\lambda)$, and*
(b) *the diagram*

$$\begin{tikzcd} & P(\lambda) & \subset & \Lambda_{\mathbb{R}} \\ X_\lambda \arrow[ur, "\mu_T"] \arrow[r, "\mu_\lambda"'] & \Delta_{\underline{w}_0}(\lambda) \arrow[u, "\pi_\lambda"'] & \subset & \mathbb{R}^n \end{tikzcd}$$

*commutes, where the vertical arrow $\pi_\lambda$ is the linear projection given in* (7.1).

7.3. **Spherical varieties.** In fact, the constructions and results in Section 7.2 can be extended to the larger class of spherical varieties. Spherical varieties are the algebraic analogues of multiplicity-free Hamiltonian spaces. More precisely, let $G$ be a connected reductive algebraic group. Let $X$ be an algebraic variety equipped with an algebraic $G$-action. Then $X$ is called *spherical* if a (and hence any) Borel subgroup of $G$ has a dense open orbit. Flag varieties $G/P_\lambda$ and $\mathbb{T}$-toric varieties are examples of spherical varieties, with respect to the actions of $G$ and $\mathbb{T}$, respectively.

Let $X$ be a normal projective spherical $G$-variety of dimension $n$ and $\mathcal{L}$ a $G$-linearized very ample line bundle. Then $X$ embeds $G$-equivariantly in the projective space $\mathbb{P}(V)$ where $V = H^0(X, \mathcal{L})^*$. Let $K$ denote a maximal compact subgroup of $G$ as in the previous section, and fix a $K$-invariant Hermitian product on $V$. This induces a Kähler metric on $\mathbb{P}(V)$ and hence on the smooth locus of $X$. With respect to the corresponding symplectic structure, (the smooth locus of) $X$ is a Hamiltonian $K$-space; let $\mu_K$ denote the moment map. The

Kirwan polytope, denoted $Q(X,\mathcal{L})$, is defined to be the intersection of the moment map image of $\mu_K$ with the positive Weyl chamber of $\mathrm{Lie}(K)^*$.

For this discussion we fix a reduced decomposition $\underline{w}_0$ for the longest element $w_0$. In [Ok97] and [AlBr04] the authors introduce a polytope $\Delta_{\underline{w}_0}(X,\mathcal{L}) \subset \Lambda^+_{\mathbb{R}} \times \mathbb{R}^N$ defined as

$$\Delta_{\underline{w}_0}(X,\mathcal{L}) := \{(\lambda, x) \mid \lambda \in Q(X,\mathcal{L}),\ x \in \Delta_{\underline{w}_0}(\lambda)\}.$$

That is, $\Delta_{\underline{w}_0}(X,\mathcal{L})$ is the polytope fibered over the moment polytope $Q(X,\mathcal{L})$ with the string polytopes as fibers. We call the polytope $\Delta_{\underline{w}_0}(X,\mathcal{L})$ the *string polytope of the spherical variety* $X$. In [AlBr04] and [Ka05] it is shown that there is a flat degeneration of $X$ to the toric variety associated to the rational polytope $\Delta_{\underline{w}_0}(X,\mathcal{L})$. In fact, in [Ka11] it is shown that, with respect to certain choices of valuation, the string polytopes $\Delta_{\underline{w}_0}(\lambda)$ and $\Delta_{\underline{w}_0}(X,\mathcal{L})$ can be realized as Okounkov bodies, and the degenertions in [Ca02], [AlBr04] and [Ka05] all fit into the general toric degeneration framework discussed in Section 4.

Generalizing the case of the flag varieties discussed in Section 7.2, we have the following. Let $P(X,\mathcal{L})$ denote the polytope which is the convex hull of the $W$-orbit of the Kirwan polytope $Q(X,\mathcal{L})$. It can be shown that $P(X,\mathcal{L})$ is precisely the moment polytope for the Hamiltonian $T$-action on $X$. Let $\mu_T$ denote the moment map for this action. Applying Theorem 5.20 to $(X,\mathcal{L})$, we obtain the following.

**Theorem 7.3.** *Let $X$ and $\mathcal{L}$ be as above. Let $\dim_{\mathbb{C}} X = n$. Then there exists an integrable system $\mu = (F_1, \ldots, F_n) : X \to \mathbb{R}^n$ such that*

(a) *the image of $\mu$ can be identified with the string polytope $\Delta_{\underline{w}_0}(X,\mathcal{L})$, and*
(b) *the diagram*

$$\begin{tikzcd} & P(X,\mathcal{L}) \\ X \arrow[ur, "\mu_T"] \arrow[r, "\mu"'] & \Delta_{\underline{w}_0}(X,\mathcal{L}) \arrow[u, "\pi"'] \end{tikzcd}$$

*commutes, where the vertical arrow $\pi$ is the linear projection given in* (7.1).

**Remark 7.4.** Let $X$ be a smooth spherical $G$-variety of dimension $n$ equipped with a $G$-linearized very ample line bundle $\mathcal{L}$. It can be shown that the space of smooth $K$-invariant functions on $X$ is commutative, in the sense that any two smooth $K$-invariant functions Poisson-commute. We believe that the integrable system $\mu = (F_1, \ldots, F_n)$ in Theorem 7.3 can be constructed so that $(F_1, \ldots, F_r)$ are smooth $K$-invariant functions on all of $X$, where $r \leq n$ is the so-called *rank* of the spherical variety (i.e., the minimal codimension of a $K$-orbit).

7.4. **Weight varieties.** We maintain the notation of Section 7.2. Let $P(\lambda)$ denote the convex hull of the $W$-orbit of $\lambda$. Recall that a **weight variety** $X_{\lambda,\gamma}$ is the GIT quotient of $X_\lambda$ by the action of $\mathbb{T}$ twisted by an integral weight $\gamma \in P(\lambda) \cap \Lambda^+$. More precisely, let $\gamma \in P(\lambda) \cap \Lambda^+$ be a character of $\mathbb{T}$ and $\mathcal{L}_\lambda(-\gamma)$ be the $\mathbb{T}$-line bundle $\mathcal{L}_\lambda$ where the action of $\mathbb{T}$ is twisted by $-\gamma$. Then we define

$$X_{\lambda,\gamma} := X_\lambda^{ss}(\mathcal{L}_\lambda(-\gamma))/\mathbb{T}.$$

An alternative description is as follows. Let $V_\lambda^{(\gamma)}$ denote the $\gamma$-weight space in the $G$-module $V_\lambda$. The weight variety $X_{\lambda,\gamma}$ is $\mathrm{Proj}(R_\lambda^{(\gamma)})$ where

$$R_\lambda^{(\gamma)} := \bigoplus_k (V_{k\lambda}^*)^{(k\gamma)}.$$

In other words, $R_\lambda^{(\gamma)}$ is the $\mathbb{T}$-invariant subalgebra of $R(L_\lambda)$ for the $(-\gamma)$-twisted action of $\mathbb{T}$ on $R(L_\lambda)$, defined by

$$t *_\gamma f := \gamma(t)^{-k}(t \cdot f) \quad \forall t \in \mathbb{T},\ \forall f \in L^k.$$

When $\gamma$ is a regular value for the moment map $\mu_T$, and $T$ acts freely on $\mu_T^{-1}(0)$, the variety $X_{\lambda,\gamma}$ can also be identified with the symplectic quotient of $X_\lambda$ at the value $\gamma$.

Using Theorem 6.8 we can now recover the following theorem of Foth and Hu [FoHu05].

**Theorem 7.5.** *There exists a flat degeneration of the weight variety $X_{\lambda,\gamma}$ to a projective toric variety $X_{\lambda,\gamma,0}$ corresponding to the polytope*

$$\Delta_{\underline{w}_0}(\lambda,\gamma) = \Delta_{\underline{w}_0}(\lambda) \cap \pi_\lambda^{-1}(\gamma),$$

*obtained by slicing the string polytope $\Delta_{\underline{w}_0}(\lambda)$ at $\gamma$. Here the projection $\pi_\lambda$ is that given in* (7.1).

As in Section 6, the GIT quotient $X_{\lambda,\gamma}$ inherits a Kähler structure from $X_\lambda$. Let $n' = \dim_{\mathbb{C}} X_{\lambda,\gamma}$. From Theorem 6.11 and Corollary 7.2 we obtain the following.

**Corollary 7.6.** *Under the assumptions and notation as above, there exists an integrable system $\mu_{\lambda,\gamma} = (F_1, \ldots, F_{n'})$ on $X_{\lambda,\gamma}$ such that*

(a) *the image of $\mu_{\lambda,\gamma}$ is the string polytope $\Delta_{\underline{w}_0}(\lambda,\gamma)$, and*
(b) *the diagram*

(7.2)
$$\begin{tikzcd}
 & P(\lambda) & \subset & \mathrm{Lie}(T)^* \\
X_\lambda \arrow[ur, "\mu_T"] \arrow[r, "\mu_\lambda"] & \Delta_{\underline{w}_0}(\lambda) \arrow[u, "\pi_\lambda"'] & \subset & \mathbb{R}^n \\
\mu_T^{-1}(\gamma) \arrow[u, hook, "i"] \arrow[d, "p"] & & & \\
X_{\lambda,\gamma} \arrow[r, "\mu_{\lambda,\gamma}"] & \Delta_{\underline{w}_0}(\lambda,\gamma) \arrow[uu, hook] & \subset & \mathbb{R}^{n'}
\end{tikzcd}$$

*commutes, where the map $\mathbb{R}^{n'} \hookrightarrow \mathbb{R}^n$ is the inclusion of the first $n'$ coordinates.*

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